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Simplified formulas for the mean and variance of linear stochastic differential equations

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ABSTRACT

Explicit formulas for the mean and variance of the solutions of stochastic differential equations with linear drift and diffusion coefficients in state and time are derived in terms of an exponential matrix. This result improved a previous one by means of which the mean and variance are expressed in terms of a linear combination of higher dimensional exponential matrices.

1. Introduction

For their simplicity and importance, linear Stochastic Differential Equations (SDEs) have been the focus of intensive researches resulting in a broad and deep knowledge of the properties of their solutions. Among these properties, the mean and variance of the solutions have been well studied. Specifically, the Ordinary Differential Equations (ODEs) that describe the dynamics of the mean and variance are well known (see, e.g., [1]). However, since the explicit solutions of these ODEs are not available in general, numerical solutions are required. Typically, these approximate solutions can be computed by means of a numerical integrator for the differential equations or by a numerical quadrature applied to the integral representation of the mean and variance (see, e.g., [2,3]). Alternatively, for homogeneous (no affine) linear SDEs, the mean and variance could be approximated via truncated Magnus expansions (see, e.g., [4]). Nevertheless, for some subclasses of linear equations, explicit expressions for the mean and variance of the solution can be obtained, which is clearly profitable from theoretical and practical viewpoint. Example of them are the formulas for the scalar linear SDEs [1], the multidimensional linear SDEs with additive noise [5], the homogeneous linear SDEs [4],

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and the SDEs with linear drift and diffusion coefficients in state and time (see Eq. (1)) [6,7]. In particular, the explicit formulas for the mean and variance of the last class of equations have become important in the practical implementation of suboptimal linear filters [6], Local Linearization filters [7] and approximate Innovation estimators [8]. In a variety of applications, these methods have shown high effectiveness and efficiency for the estimation of unobserved components and unknown parameters of nonlinear SDEs given a set of discrete observations. Remarkable is the identification, from actual data, of neurophysiological, financial and molecular models among others (see, e.g., [9–11]). Since the formulas for the mean and variance obtained in [6,7] are expressed in terms of a linear combination of seven exponential matrices, any simplification of them might imply a sensible reduction of the computational cost of the mentioned system identification methods and, consequently, a positive impact in applications.

In this paper, simplified explicit formulas for the mean and variance of SDEs with linear drift and diffusion coefficients in state and time are obtained in terms of just one exponential matrix of lower dimensionality. The formulas are derived from the solution of the ODEs that describe the evolution of the mean and the second moment of the SDEs. The variance is then obtained from the well-known formula that involves the first two moments. The computational benefits of the simplified formulas are pointed out.

2. Notation and preliminaries

Let us consider the d-dimensional linear stochastic differential equation

$$d\mathbf{x}(t) = (\mathbf{A}\mathbf{x}(t) + \mathbf{a}(t))dt + \sum_{i=1}^{m}(\mathbf{B}_i\mathbf{x}(t) + \mathbf{b}_i(t))d\mathbf{w}^i(t) \tag{1}$$

for all $t \in [t_0, T]$, where $\mathbf{w} = (\mathbf{w}^1, \ldots, \mathbf{w}^m)$ is an m-dimensional standard Wiener process, $\mathbf{A}$ and $\mathbf{B}_i$ are $d \times d$ matrices, and $\mathbf{a}(t) = \mathbf{a}_0 + \mathbf{a}_1 t$ and $\mathbf{b}_i(t) = \mathbf{b}_{i,0} + \mathbf{b}_{i,1}t$ are d-dimensional vectors. Suppose that there exist the first two moments of $\mathbf{x}$ for all $t \in [t_0, T]$.

The ordinary differential equations for the d-dimensional vector mean $\mathbf{m}_t = E(\mathbf{x}(t))$ and the $d \times d$ matrix second moment $\mathbf{P}_t = E(\mathbf{x}(t)\mathbf{x}^\intercal(t))$ of $\mathbf{x}(t)$ are [1]

$$\frac{d\mathbf{m}_t}{dt} = \mathbf{A}\mathbf{m}_t + \mathbf{a}(t) \quad \text{and} \quad \frac{d\mathbf{P}_t}{dt} = \mathbf{A}\mathbf{P}_t + \mathbf{P}_t\mathbf{A}^\intercal + \sum_{i=1}^{m}\mathbf{B}_i\mathbf{P}_t\mathbf{B}_i^\intercal + \mathcal{B}(t),$$

where

$$\mathcal{B}(t) = \mathbf{a}(t)\mathbf{m}_t^\intercal + \mathbf{m}_t\mathbf{a}^\intercal(t) + \sum_{i=1}^{m}\mathbf{B}_i\mathbf{m}_t\mathbf{b}_i^\intercal(t) + \mathbf{b}_i\mathbf{m}_t^\intercal\mathbf{B}_i^\intercal(t) + \mathbf{b}_i(t)\mathbf{b}_i^\intercal(t). \tag{2}$$

The solution of these equations can be written as [6]

$$\mathbf{m}_t = \mathbf{m}_0 + \mathbf{L}\mathbf{e}^{\mathbf{C}(t-t_0)}\mathbf{r} \tag{3}$$

and

$$vec(\mathbf{P}_t) = e^{\mathcal{A}(t-t_0)}(vec(\mathbf{P}_0) + \int_0^{t-t_0} e^{-\mathcal{A}s}vec(\mathcal{B}(s+t_0))ds), \tag{4}$$

where $\mathbf{m}_0 = E(\mathbf{x}(t_0))$ and $\mathbf{P}_0 = E(\mathbf{x}(t_0)\mathbf{x}^\intercal(t_0))$ are the first two moments of $\mathbf{x}$ at t_0, and the matrices $\mathbf{C}$, $\mathbf{L}$ and $\mathbf{r}$ are defined as

$$\mathbf{C} = \begin{bmatrix} \mathbf{A} & \mathbf{a}_1 & \mathbf{A}\mathbf{m}_0 + \mathbf{a}(t_0) \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \in \Re^{(d+2)\times(d+2)}, \tag{5}$$