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Existence of positive solutions for a class of discrete Dirichlet boundary value problems

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ABSTRACT

Let $x = f(x)$, where $x \in R$. Clearly, if there exists $b > a > 0$ such that $f \in C[a, b]$ and either $f(a) \le a$ and $f(b) \ge b$ or $f(a) \ge a$ and $f(b) \le b$, then there is $x^* \in [a, b]$ such that $x^* = f(x^*)$, that is, the function $f(x)$ has a fixed point $x^* \in [a, b]$. By using the above main idea and famous Guo–Krasnosel'skii fixed point theorem, existence of positive solutions for a nonlinear second order difference equation and a discrete second order system with the Dirichlet boundary conditions will be considered. The new existence results will be obtained. In particular, the main idea is also valid for the partial difference problems or the general nonlinear algebraic system.

1. Introduction

Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a cone if it satisfies the following two conditions: (i) $x \in P$ and $\lambda \ge 0$ implies that $\lambda x \in P$, and (ii) $x \in P$ and $-x \in P$ implies that $x = \theta$, where $\theta \in E$ is called to be the zero element of E.

Now we state the Guo–Krasnosel'skii fixed point theorem concerning cone expansion and compression of norm type as follows (see [1,2]).

Lemma 1. *Let Ω_1 and Ω_2 be two bounded open sets in E such that $\theta \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Suppose that $A : P \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right) \to P$ is completely continuous. If either* (H_1) $\|Ax\| \le \|x\|$ *for $x \in P \cap \partial\Omega_1$ and $\|Ax\| \ge \|x\|$ for $x \in P \cap \partial\Omega_2$, or* (H_2) $\|Ax\| \ge \|x\|$ *for $x \in P \cap \partial\Omega_1$ and $\|Ax\| \le \|x\|$ for $x \in P \cap \partial\Omega_2$ holds, then A has at least one fixed point in $P \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$.*

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By using Lemma 1, many existence results, in particular, existence of positive solutions for the boundary value problems, the periodic systems, the nonlinear integral equation of Hammerstein type, etc. have been extensively established. In the general case, the growth conditions of the nonlinear term will be given at the neighborhood of the zero point and the infinite point. In view of Lemma 1, theoretically we may obtain any fixed point in Banach space E when we structure a suitable cone P and two suitable open sets Ω_1 and Ω_2.

For example, we let $E = R$ and consider the existence of roots for the equation $x = f(x)$. In this case, there are only two cones $R_+ = [0, +\infty)$ and $R_- = (-\infty, 0]$. Clearly, if there exists $b > a > 0$ such that $f \in C[a, b]$ and either $f(a) \leq a$ and $f(b) \geq b$ or $f(a) \geq a$ and $f(b) \leq b$, then there is $x^* \in [a, b]$ such that $x^* = f(x^*)$, that is, the function $f(x)$ has a fixed point $x^* \in [a, b]$. At this time, the function only need to have definition on $[a, b]$.

Then, can we expand this good idea to a general case? In this paper, we will consider the discrete Dirichlet boundary value problem of the form

$$\begin{cases} \Delta^2 x_{i-1} + f(x_i) = 0, & i \in [1, n], \\ x_0 = 0 = x_{n+1}, \end{cases} \tag{1}$$

where n is a positive integer, $[1, n] = \{1, 2, \cdots, n\}$, Δ is the forward difference operator, that is, $\Delta x_{i-1} = x_i - x_{i-1}$, and $\Delta^2 x_{i-1} = \Delta(\Delta x_{i-1})$. By constructing some suitable P and the suitable open sets Ω_1 and Ω_2, the existence of positive solutions of (1) will be considered.

Let $x = \operatorname{col}(x_1, x_2, \ldots, x_n)$, $f(x) = \operatorname{col}(f(x_1), f(x_2), \ldots, f(x_n))$, and

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ & \cdots & \cdots & \cdots & \\ 0 & \cdots & & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}_{n \times n}$$

which has an inverse $G = (g_{ij})$, which is given by

$$g_{ij} = \begin{cases} \dfrac{(n-i+1)j}{n+1}, & 1 \leq j \leq i \leq n, \\ \dfrac{(n-j+1)i}{n+1}, & 1 \leq i \leq j \leq n. \end{cases} \tag{2}$$

See Cheng [3]. Then problem (1) can be rewritten by matrix and vector in the form

$$Ax = f(x), \tag{3}$$

$$x = Gf(x) \tag{4}$$

or

$$x_i = \sum_{j=1}^{n} g_{ij} f(x_j) \quad \text{for } i \in [1, n]. \tag{5}$$

Clearly, the nonlinear algebraic system (4) can be regarded as an operator equation in the Banach space $E = R^n$. Thus, we can use the Guo–Krasnosel'skii fixed point theorem to obtain some existence results of positive solutions for (4). Such problem will be considered in the next section.

Our idea can also be used to consider the difference system of the form

$$\begin{cases} \Delta^2 x_{i-1} + f(x_i, y_i) = 0, & i \in [1, n], \\ \Delta^2 y_{i-1} + g(x_i, y_j) = 0, & i \in [1, n], \\ x_0 = x_{n+1} = y_0 = y_{n+1} = 0. \end{cases} \tag{6}$$