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Global existence of periodic solutions in an infection model[☆]

Xinguo Sun[a,b], Junjie Wei[a,*]

[a] *Department of Mathematics, Harbin Institute of Technology, Harbin, Heilongjiang, 150001, PR China*
[b] *School of Science, China University of Petroleum (East China), Qingdao, 266580, PR China*

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ABSTRACT

In this paper, a HTLV-I infection model with CTL immune response is considered. Taking the immune delay as a bifurcation parameter we investigate the global existence of periodic solutions of this model which shows existence of multiple periodic solutions theoretically.

1. Introduction

Li and Shu [1] have considered the following HTLV-I model

$$\begin{cases} x'(t) = \lambda - d_1 x(t) - \beta x(t) y(t), \\ y'(t) = \beta x(t) y(t) - d_2 y(t) - \gamma y(t) z(t), \\ z'(t) = \mu y(t-\tau) z(t-\tau) - d_3 z(t). \end{cases} \tag{1.1}$$

The numbers of uninfected CD4$^+$ T-cell population and infected CD4$^+$ T-cell population are denoted by $x(t)$ and $y(t)$, respectively. $z(t)$ denotes the number of HTLV-I-specific CD8$^+$ CTLs. For more detail about the model, we refer readers to [1–3].

For system (1.1), Li and Shu [1] have obtained the global stability of equilibria and local bifurcation. The main goal of this paper is to study global existence of periodic solutions of this system. Our results obtained are a complement of the works of Li and Shu [1]. On the global existence of periodic solutions for delay differential equations, we refer readers to [4,5,7–10].

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* Corresponding author.
E-mail address: weijj@hit.edu.cn (J. Wei).

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We state some results of Li and Shu [1] in following which shall be used.

For $\tau > 0$, let $\mathcal{C} = \mathcal{C}([-\tau, 0], \mathbb{R})$, and the nonnegative cone of $\mathcal{C}$ is defined as $\mathcal{C}^+ = \mathcal{C}([-\tau, 0], \mathbb{R}_+)$. Initial conditions for system (1.1) are chosen at $t = 0$ as

$$\varphi \in \mathbb{R}_+ \times \mathcal{C}^+ \times \mathcal{C}^+, \quad \varphi = (\varphi_1, \varphi_2, \varphi_3), \qquad \varphi_i(0) > 0, \quad i = 1, 2, 3. \tag{1.2}$$

Lemma 1.1. *Under initial condition (1.2), all solutions of system (1.1) are positive and ultimately bounded in $\mathbb{R}_+ \times \mathcal{C} \times \mathcal{C}$. Furthermore, all solutions eventually enter and remain in the following bounded region:*

$$\Gamma^* = \left\{ (x, y, z) \in \mathbb{R}_+ \times \mathcal{C}^+ \times \mathcal{C}^+ : \| x \| \le \frac{\lambda}{d_1} + \varepsilon, \| x + y \| \le \frac{\lambda}{\widetilde{d}} + \varepsilon, \| x + y + \frac{\gamma}{\mu} z \| \le \frac{\lambda}{d} + \varepsilon \right\},$$

where $\widetilde{d} = \min\{d_1, d_2\} > 0$, $d = \min\{d_1, d_2, d_3\} > 0$, ε is arbitrarily small positive number.

From Lemma 1.1, we know that the dynamics of system (1.1) can be analyzed in the following bounded feasible region

$$\Gamma = \left\{ (x, y, z) \in \mathbb{R}_+ \times \mathcal{C}^+ \times \mathcal{C}^+ : \| x \| \le \frac{\lambda}{d_1}, \| x + y \| \le \frac{\lambda}{\widetilde{d}}, \| x + y + \frac{\gamma}{\mu} z \| \le \frac{\lambda}{d} \right\}.$$

Moreover, the region Γ is positive invariant for system (1.1).

System (1.1) always has an infection-free equilibrium $P_0 = (x_0, 0, 0), x_0 = \frac{\lambda}{d_1}$. In addition to P_0, the system can have two chronic-infection equilibria $P_1 = (\overline{x}, \overline{y}, 0)$ and $P_2 = (x^*, y^*, z^*)$ in Γ.

Let

$$R_0 = \frac{\lambda \beta}{d_1 d_2}, \qquad R_1 = \frac{\lambda \beta \mu}{d_1 d_2 \mu + \beta d_2 d_3}. \tag{1.3}$$

They are called the basic reproduction numbers for viral infection and for CTL response, respectively. Obviously $R_1 < R_0$ always holds.

It can be verified that the equilibrium $P_1 = (\overline{x}, \overline{y}, 0)$ exists if and only if $R_0 > 1$ and that

$$\overline{x} = \frac{d_2}{\beta} = \frac{\lambda}{d_1 R_0}, \qquad \overline{y} = \frac{\lambda \beta - d_1 d_2}{\beta d_2} = \frac{d_1(R_0 - 1)}{\beta}. \tag{1.4}$$

The coordinates of the equilibrium $P_2 = (x^*, y^*, z^*)$ are given by

$$x^* = \frac{\lambda \mu}{d_1 \mu + \beta d_3} = \frac{d_2 R_1}{\beta}, \qquad y^* = \frac{d_3}{\mu}, \qquad z^* = \frac{\beta \lambda \mu - d_1 d_2 \mu - \beta d_2 d_3}{(d_1 \mu + \beta d_3)\gamma} = \frac{d_1 d_2 \mu + \beta d_2 d_3}{(d_1 \mu + \beta d_3)\gamma}(R_1 - 1). \tag{1.5}$$

Therefore, P_2 exists in the interior of Γ if and only if $R_1 > 1$.

Lemma 1.2. (1) *If $R_1 < 1 < R_0$, then the equilibrium P_1 is globally asymptotically stable in $\Gamma \setminus \{x\text{-axis}\}$. If $R_1 > 1$, then P_1 is unstable.*

(2) *If $R_1 > 1$, then the equilibrium P_2 is globally asymptotically stable when $\tau = 0$.*

2. Hopf branches analysis

In this section, we always assume that $R_1 > 1$ holds. The characteristic equation associated with the linearization of system (1.1) at P_2 is given by

$$\xi^3 + a_2 \xi^2 + a_1 \xi + a_0 + (b_2 \xi^2 + b_1 \xi + b_0) e^{-\tau \xi} = 0. \tag{2.1}$$