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Asymptotic integration of second-order nonlinear delay differential equations

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ABSTRACT

We study the asymptotic integration problem for second-order nonlinear delay differential equations of the form $(p(t)x'(t))' + q(t)x(t) = f(t, x(g(t)))$. It is shown that if u and v are principal and nonprincipal solutions of equation $(p(t)x')' + q(t)x = 0$, then there are solutions $x_1(t)$ and $x_2(t)$ of the above nonlinear equation such that $x_1(t) = au(t) + o(u(t))$, $t \to \infty$ and $x_2(t) = bv(t) + o(v(t))$, $t \to \infty$.

1. Introduction

The problem of asymptotic integration for second-order differential equations has received considerable attention during the last decades. Most results however are related to differential equations without delay, see the excellent survey paper by Agarwal et al. [1] and the references cited therein. For some recent results we may refer the author in particular to [2–4].

In the present work we study the effect of delay on the asymptotic integration problem for the second-order nonlinear delay differential equation

$$(p(t)x'(t))' + q(t)x(t) = f(t, x(g(t))), \tag{1}$$

where $p \in C([t_0, \infty), [0, \infty))$, $q \in C([t_0, \infty), \mathbb{R})$, $f \in C([t_0, \infty) \times \mathbb{R}, \mathbb{R})$, and $g \in C([t_0, \infty), \mathbb{R})$ such that $\lim_{t\to\infty} g(t) = \infty$ and $-\tau \le g(t) \le t$ for some $\tau > 0$.

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To the best of our knowledge the first studies in this direction were done by Sobol' [5], Cooke [6–8], Wong [9], Onose [10], and Kusano and Naito [11].

In [11, Theorem 2] the authors showed the existence of a solution of

$$(p(t)x'(t))' = f(t, x(g(t))) \tag{2}$$

that is asymptotic to $a\rho(t)$, $a \neq 0$ as $t \to \infty$, where ρ satisfies $\int^{\infty} \rho(g(t)) f(t, c^2(\rho(g(t)))^2)dt < \infty$ for some $c > 0$, and $f(t, x)$ is sub-linear or super-linear.

Trench [12] studied the more general equation (1) and proved that the equation has a solution $x(t)$ which behaves for large t like a given solution $y(t)$ of the ordinary differential equation

$$(p(t)x'(t))' + q(t)x(t) = 0. \tag{3}$$

More precisely, he obtained $x(t) = y(t) + o(y_i(t))$, $t \to \infty$, $i = 1, 2$, where y_i might be a principal or a nonprincipal solution of (3).

In 2007, Kusano et al. [2] investigated the asymptotic integration problem for

$$(p(t)x')' + q(t)x = f(t, x), \quad t \geq t_0 \tag{4}$$

under the condition $|f(t, x)| \leq h(t)\phi(x)$, where h is a continuous function on $[a, \infty)$ for some $a > 0$, and ϕ is a continuous and non-decreasing function on $[0, \infty)$. They proved that if u is a principal solution of (3) such that for some $b > 0$,

$$\int_a^{\infty} \frac{1}{p(t)u^2(t)} \int_t^{\infty} h(s)u(s)\phi(bu(s))dsdt < \infty$$

holds, then there exists a positive solution $x(t)$ of the nonlinear equation (4) such that

$$x(t) = (b/2)u(t) + o(u(t)), \quad t \to \infty.$$

A time scale extension of this theorem can be found in Ünal and Zafer [13].

Very recently, Ertem and Zafer [14] proved that if there exist $\phi \in C([0, \infty), [0, \infty))$ and h_1, $h_2 \in C([t_1, \infty), [0, \infty))$ satisfying

$$|f(t, x)| \leq h_1(t)\phi\left(\frac{|x|}{v(t)}\right) + h_2(t), \quad t \geq T \tag{5}$$

for some $T > t_1$, and if $\int_T^{\infty} v(s)h_i(s)ds < \infty$ for $i = 1, 2$, then for any given $a, b \in \mathbb{R}$ there is a solution $x(t)$ of (4) such that

$$x(t) = av(t) + bu(t) + o(u(t)), \quad t \to \infty.$$

In the present paper we provide a contribution to the asymptotic integration problem for delay equations of the form (1). Our motivation stems from the above mentioned works and a compactness criterion employed in [15] for the study of equations without delay. We will observe that the delay is very crucial in changing the behavior of the solutions.

2. Main results

Recall that if Eq. (3) is nonoscillatory at ∞, then there exist such solutions u and v, called principal and nonprincipal respectively. While a principal solution u is unique up to a constant multiple, any solution v that is linearly independent of u is a nonprincipal solution (see e.g., [16,17]).

In the proofs we will take the following pair of principal and nonprincipal solutions:

$$\left\{u(t),\, v(t) = u(t)\int_{t_1}^{t} \frac{1}{p(s)u^2(s)}\,ds\right\} \tag{6}$$