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On inverse problem for a convolution integro-differential operator with Robin boundary conditions

S.A. Buterin[a,*], A.E. Choque Rivero[b]

[a] Department of Mathematics, Saratov University, Astrakhanskaya 83, Saratov 410012, Russia

[b] Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo. Edificio C3A, Cd. Universitaria. C. P. 58040 Morelia, Mich., Mexico

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ABSTRACT

We study the inverse spectral problem of recovering a second-order integro-differential operator with a convolution component and Robin boundary conditions from its spectrum. We prove the uniqueness theorem and that the standard spectral asymptotics is a necessary and sufficient condition for solvability of the inverse problem in an appropriate class. The proof is constructive and gives an algorithm for solving the inverse problem.

1. Introduction

Let $\{\lambda_k\}_{k\geq 0}$ be the spectrum of the boundary value problem $L = L(M, h, H)$ of the form

$$\ell y := -y'' + \int_0^x M(x-t)y'(t)\,dt = \lambda y, \quad 0 < x < \pi, \tag{1}$$

$$U(y) := y'(0) - hy(0) = 0, \qquad V(y) := y'(\pi) + Hy(\pi) = 0, \tag{2}$$

where λ is the spectral parameter, $M(x)$ is a complex-valued function, $(\pi - x)M(x) \in L_2(0, \pi)$ and $h, H \in \mathbb{C}$. By the standard method involving Rouché's theorem one can prove that the eigenvalues λ_k have the form

$$\lambda_k = (k + \varkappa_k)^2, \quad \{\varkappa_k\} \in l_2. \tag{3}$$

We study an inverse spectral problem for L. Inverse problems of spectral analysis consist in recovering operators from given their spectral characteristics. Such problems often appear in mathematics, mechanics,

* Corresponding author.
E-mail addresses: buterinsa@info.sgu.ru (S.A. Buterin), abdon@ifm.umich.mx (A.E. Choque Rivero).

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physics, electronics, geophysics, meteorology and other branches of natural sciences and engineering. The greatest success in the inverse problem theory has been achieved for the Sturm–Liouville operator $\ell_1 y := -y'' + q(x)y$ (see, e.g., [1–4]) and afterwards for higher-order differential operators [5–7]. For example, it is known that the potential $q(x)$ can be uniquely determined by specifying the spectra of two boundary value problems for equation $\ell_1 y = \lambda y$ with one common boundary condition.

For integro-differential and other classes of nonlocal operators inverse problems are more difficult for investigation, and the classical methods either are not applicable to them or require essential modifications (see [4,8–17] and the references therein). The first substantive research of an inverse problem for integro-differential operators was carried out in [8], where a perturbation of the Sturm–Liouville operator with Dirichlet boundary conditions by the Volterra convolution operator was considered. It was proved that the specification of only the spectrum uniquely determines the convolution component. Moreover, developing the idea of Borg's method a constructive procedure for solving this inverse problem was obtained along with the local solvability and stability. In [13] the global solvability was proved by reducing this inverse problem to solving the so-called main nonlinear integral equation, which was solved globally. Earlier by a particular case of this approach the analogous results were obtained for the operator (1) with Dirichlet boundary conditions [11].

Robin boundary conditions (2) bring additional difficulties in studying the inverse problem for L. In the present paper we start the study with the following problem.

Inverse Problem 1. Given $\{\lambda_k\}_{k\geq 0}$ and h, H, find $M(x)$.

For this inverse problem we prove the following uniqueness theorem.

Theorem 1. *The specification of the spectrum $\{\lambda_k\}_{k\geq 0}$ uniquely determines the function $M(x)$, provided that the coefficients h, H are known a priori.*

We note that Inverse Problem 1 is overdetermined. Indeed the spectrum possesses also some information on the coefficients of the boundary conditions. In particular, we prove that if $h = 0$, then along with $M(x)$ the coefficient H is also determined, i.e. the uniqueness theorem holds for the following problem.

Inverse Problem 2. Given $\{\lambda_k\}_{k\geq 0}$ and $h = 0$, find H and $M(x)$.

Moreover, the following theorem holds.

Theorem 2. *For arbitrary complex numbers λ_k, $k \geq 0$, of the form (3) there exists a unique (up to values on a set of measure zero) function $M(x)$, $(\pi - x)M(x) \in L_2(0, \pi)$, and a unique number $H \in \mathbb{C}$, such that $\{\lambda_k\}_{k\geq 0}$ is the spectrum of the problem $L(M, 0, H)$.*

Thus, the asymptotics (3) is a necessary and sufficient condition for the solvability of Inverse Problem 2. The importance of the assumption $h = 0$ is explained in Remark 2 (see Section 3). We leave open whether the analogous criterium can be obtained assuming only that h is known a priori but $h \neq 0$, or symmetrically: H is given while h is unknown.

In the next section we derive the main nonlinear equation and prove its global solvability. In Section 3 we provide the proof of Theorems 1 and 2, which is constructive and gives algorithms for solving the inverse problems (Algorithms 1 and 2).

2. Main nonlinear integral equation

Let $C(x, \lambda), S(x, \lambda), \varphi(x, \lambda)$ be solutions of Eq. (1) satisfying the initial conditions

$$C(0, \lambda) = S'(0, \lambda) = \varphi(0, \lambda) = 1, \qquad C'(0, \lambda) = S(0, \lambda) = 0, \qquad \varphi'(0, \lambda) = h.$$