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A new Picone's dynamic inequality on time scales with applications

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ABSTRACT

In this paper, we will derive a new dynamic Picone-type inequality for half-linear dynamic equations and dynamic inequalities of second order on an arbitrary time scale $\mathbb{T}$. As a consequence, we will apply this new Picone inequality to get a new Wirtinger-type inequality on time scales with two different weighted functions. The results contain the Wirtinger inequalities formulated by Beesack, Lee and Jaroš for the continuous case. For the discrete case our results are also new.

1. Introduction

In 1910 Picone [1] derived the following identity

$$\frac{d}{dx}\left[\frac{u}{v}\left(vp_1u'-up_2v'\right)\right]=u\left(p_1u'\right)'-\frac{u^2}{v}\left(p_2v'\right)'+\left(p_1-p_2\right)\left(u'\right)^2+p_2\left(u'-\frac{u}{v}v'\right)^2, \tag{1.1}$$

where u, v, p_1u', p_2v' are differentiable functions with respect to x and $v(x)\neq 0$ for $x\in\mathbb{I}\subset\mathbb{R}$. This identity was discovered by Picone during his attempt to introduce a generalization of the famous Sturm Comparison Theorem for the formally self-adjoint second order linear differential equations

$$(p_1(x)u'(x))'+q_1(x)u(x)=0, \qquad (p_2(x)v'(x))'+q_2(x)v(x)=0. \tag{1.2}$$

The classical Picone identity was a very useful tool in the development of the classical Sturmian theory. Much work has been done and many papers which deal with various generalizations and extensions, even

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for second order half-linear differential equations with p-Laplacian, have appeared in the literature, we refer the reader to the book [2] and the papers [3,4]. These various generalizations have also been extended to difference equations (see [5,6] and the references cited therein).

One of the main applications of identity (1.1) is the investigation of Wirtinger type inequalities which may be used and applied in obtaining lower bounds of the eigenvalues of the eigenvalue problems of second order ordinary differential equations (see [7–9]).

In the following, we recall some of the related work that motivates and explains the aim of our paper. As a connection between Wirtinger type inequalities and solutions of differential equations, Beesack [10] considered a solution $y_1(x)$ of the following second-order linear differential equation

$$y''(x) + p(x)y(x) = 0 \quad \text{for } x \in [0, a],$$

with $y_1(0) = 0$, $y_1'(0) \geq 0$, $y_1(x) > 0$, where $p(x)$ is assumed to be continuous on $[0, a]$ and proved that if $f' \in L^2$ with $f(0) = 0$, then

$$\int_0^a (f'(x))^2 \, dx \geq \int_0^a p(x) (f(x))^2 \, dx. \tag{1.3}$$

Moreover, equality holds if and only if $f(x) = Ay_1(x)$. Lee [9] considered a more general differential equation to establish a generalized Wirtinger-type inequality. In particular, he considered the following differential operator

$$Lv := \left(p \left|v'\right|^{\alpha-1} v'\right)' + q \left|v\right|^{\alpha-1} v,$$

for x in the interval $\mathbb{I} = [a, b] \subseteq \mathbb{R}$ and v is a positive solution for the following differential inequality

$$-Lv \geq \lambda_0 rv, \quad x \in \mathbb{I},$$

to obtain the validity of the following Wirtinger-type inequality

$$\int_{\mathbb{I}} (q \left|v\right|^{\alpha-1} + \lambda_0 r) u^2 dx \leq \int_{\mathbb{I}} p \left(u'\right)^2 \left|v'\right|^{\alpha-1} dx + S_1(u, v) - S_2(u, v), \tag{1.4}$$

for every function $u \in AC(\mathbb{I})$, i.e., the set of all real-valued functions which are absolutely continuous on every closed subinterval of $\mathbb{I}$, where

$$S_1(u, v) := \lim_{x \to a^+} \frac{p(x)u^2(x) \left|v'(x)\right|^{\alpha-1}}{v(x)}, \qquad S_2(u, v) := \lim_{x \to b^-} \frac{p(x)u^2(x) \left|v'(x)\right|^{\alpha-1}}{v(x)}.$$

Recently in 2011 Jaroš [11] extended his results in [3] and made use of the close connection between Wirtinger-type inequalities with Euler–Lagrange differential equations associated with variational problems to prove some new Wirtinger-type inequalities. In particular, Jaroš considered the following differential inequality

$$\left(p \left|v'\right|^{\alpha-1} v'\right)' + q \left|v\right|^{\alpha-1} v + \lambda r \left|v\right|^{\beta-1} v \leq 0, \quad \text{in } \mathbb{I} = [a, b], \tag{1.5}$$

and proved that

$$\int_{\mathbb{I}} (q \left|v\right|^{\alpha-\beta} + \lambda r) \left|u\right|^{\beta+1} dx \leq \int_{\mathbb{I}} p \left|u'\right|^{\beta+1} \left|v'\right|^{\alpha-\beta} dx + S_a(u, v) - S_b(u, v), \tag{1.6}$$

where

$$\left.\begin{aligned} S_a(u, v) &:= \lim_{x \to a^+} \frac{p(x) \left|u\right|^{\beta+1}(x) \left|v'(x)\right|^{\alpha-1} v'(x)}{\left|v\right|^{\beta-1} v}, \\ S_b(u, v) &:= \lim_{x \to b^-} \frac{p(x) \left|u\right|^{\beta+1}(x) \left|v'(x)\right|^{\alpha-1} v'(x)}{\left|v\right|^{\beta-1} v}, \end{aligned}\right\}$$