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Heteroclinic cycles in a class of 3-dimensional piecewise affine systems

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ABSTRACT

It is well known that homoclinic and heteroclinic cycles can potentially result in chaos in dynamical systems. However, it is not easy to find the homoclinic or heteroclinic cycles in concrete systems. Therefore, how to prove the existence of homoclinic and heteroclinic cycles is an important problem in modern dynamical systems. In this paper, for a class of 3-dimensional piecewise affine systems we present succinct sufficient conditions for the existence of three types of heteroclinic cycles by mathematical analysis. As applications, two existence results of chaotic invariant sets are obtained. In addition, some examples are presented to illustrate our results.

1. Introduction

Homoclinic and heteroclinic cycles are playing an important role in the study of chaotic dynamics, because a lot of chaotic phenomena can be associated with homoclinic and heteroclinic cycles. For example, the famous Smale–Birkhoff theorem showed that, for a 2-dimensional diffeomorphism, transverse homoclinic orbits or heteroclinic cycles imply the existence of horseshoe [1,2]. For differential vector fields, the famous Shil'nikov theorems showed that the existence of a homoclinic or a heteroclinic cycle implies the existence of a countable number of horseshoes in a neighborhood of this cycle under some conditions [2–5].

It is easy to see that for ordinary differential equations, to find the chaos of Shil'nikov type the key is to prove the existence of homoclinic or heteroclinic cycles, which has received much attention. Up to now, for some smooth systems, some methods have been developed for the existence and the construction of the homoclinic or heteroclinic cycles. For example, in [6,7], Deng first showed the specific construction of chaotic attractors with some smooth systems by singular perturbations; later, Deng and Hines in [8] proved the existence of homoclinic orbits in a concrete food-chain model, called Rosenzweig–MacArthur model, by multi-time scale analysis; in [9], Tigan and Turaev showed that the homoclinic orbits exist in the Shimizu–Morioka system by a refined version of the method of comparison systems developed in [10,11] for the analytic proof of the existence of homoclinic cycles in the Lorenz model; in [12], Shang and Han proposed an improved undetermined coefficient method to the non-existence of the homoclinic cycle for a concrete system; Recently, a rather general and effective method called the Fishing principle was presented by Leonov et al. in [13–16], which allows one to prove rigorously the existence of homoclinic orbits in various systems, such as the Lorenz system, the Shimizu–Morioka system. Furthermore, for some concrete piecewise linear systems the existence of homoclinic cycles or heteroclinic cycles has been proved by numerical calculations or analytic methods. For example, in [17], Pisarchik et al. showed that

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homoclinic orbits and homoclinic chaos can exist in a piecewise linear Rössler-like circuit by numerical simulation; in [18], Medrano-T et al. presented a general numerical method to demonstrate the existence of homoclinic cycles or heteroclinic cycles in piecewise linear double-scroll circuit; in [19], Li and Chen gave some piecewise linear chaotic systems with heteroclinic cycles by numerical simulation; in [20], an analytic proof was given for the existence of homoclinic connections and a reversible T-point heteroclinic cycle in a concrete continuous piecewise linear system by Carmona et al. Although there have been a progress as mentioned above, it is clear that more efforts are needed for finding methods to prove the existence of homoclinic or heteroclinic cycles for various systems, especially analytical methods.

In this paper, we concentrate on the existence of heteroclinic cycles in a class of 3-dimensional piecewise affine systems. Firstly we will propose two lemmas in planar systems (see Lemma 1, 2 in Section 2), and then present the main results on the existence of three types of heteroclinic cycles by these lemmas. Moreover, the main results are easy to be used in some concrete systems, such as Examples 1–3 in Section 4. Furthermore, by the methods for the proof of Shil'nikov theorems [2–5], it is easy to obtain two existence results of some chaotic invariant sets related to heteroclinic cycles in the 3-dimensional piecewise affine systems.

The rest of this paper is organized as follows. In Section 2, we recall some important definitions and prove two lemmas in planar linear systems. In Section 3, we give the main results on the existence of three types of heteroclinic cycles and their proofs by using these lemmas in Section 2. Based on the main results in this paper, we discuss the existence of chaotic invariant sets related to heteroclinic cycles in the 3-dimensional piecewise affine systems and get two consequences with one of their proofs being presented in the Appendix for readers' convenience. In Section 4, we give some examples to illustrate our main results. In Section 5, we provide the conclusions.

2. Preliminary

For convenience, we recall some important definitions and show two useful conclusions in planar linear systems in this section.

Definition (*[21]*). A *homoclinic orbit* is a trajectory $x(t)$ that connects an equilibrium x^* to itself, i.e., $x(t) \to x^*$ as $t \to \pm\infty$. A*heteroclinic orbit* is a trajectory $x(t)$ that connects two different equilibria x_1^* and x_2^*, i.e., $x(t) \to x_1^*$ as $t \to +\infty$ and $x(t) \to x_2^*$ as $t \to -\infty$.

Now consider general planar linear system as follows

$$\dot{\mathbf{x}} = \mathrm{A}_0\mathbf{x}, \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^2$. For $\mathbf{x}_0 \in \mathbb{R}^2$, we denote the positive semi-orbit of $\mathbf{x}_0$ by $O_+(\mathbf{x}_0)$, i.e.,

$$O_+(\mathbf{x}_0) = \{\Phi(t, \mathbf{x}_0)|t > 0\} = \{\exp(\mathrm{A}_0 t)\mathbf{x}_0|t > 0\}$$

where $\Phi(t, \cdot)$ denotes the flow generated by (1). Let $\mathrm{L} = \{\mathbf{x} \in \mathbb{R}^2|\mathbf{k}^T\mathbf{x} = c\}$ with $0 < c \in \mathbb{R}$ and $\mathbf{0} \neq \mathbf{k} = (k_1, k_2)^T \in \mathbb{R}^2$. Then L is a straight line not passing through the origin which divides the plane into three disjoint subsets L, L_+ and L_-, where

$$L_+ = \{\mathbf{x} \in \mathbb{R}^2|\mathbf{k}^T\mathbf{x} > c\}, \qquad L_- = \{\mathbf{x} \in \mathbb{R}^2|\mathbf{k}^T\mathbf{x} < c\}.$$

Obviously, the origin is in L_-.

Lemma 1. *For system (1), suppose that the eigenvalues of* A_0 *are given by* $\mu_{1,2} < 0$ *and* $\mathbf{x}_0 \in L$*, then*

$$O_+(\mathbf{x}_0) \subset L_- \quad \textit{if and only if} \quad \mathbf{k}^T\mathrm{A}_0\mathbf{x}_0 \le 0.$$

Proof. Here we only present the proof of Lemma 1 under the assumption that A_0 is given by $\mathrm{N}\begin{pmatrix}\mu_1 & 0\\ 0 & \mu_2\end{pmatrix}\mathrm{N}^{-1}$with N being an invertible matrix. If A_0 is given by $\mathrm{N}\begin{pmatrix}\mu_1 & 1\\ 0 & \mu_2\end{pmatrix}\mathrm{N}^{-1}$with $\mu_1 = \mu_2$, the proof is similar.

(a) *The proof of necessity*

Let $g(t) = \mathbf{k}^T\exp(\mathrm{A}_0 t)\mathbf{x}_0$. Obviously g is a smooth function with

$$g'(t) = \mathbf{k}^T\exp(\mathrm{A}_0 t)\mathrm{A}_0\mathbf{x}_0. \tag{2}$$

Moreover, $g(0) = c$ and $g(t) < c(t > 0)$ imply $g'(0) = \mathbf{k}^T\mathrm{A}_0\mathbf{x}_0 \le 0$.

(b) *The proof of sufficiency*

From (2), we have $g'(t) = \mathbf{k}^T\mathrm{N}\begin{pmatrix}e^{\mu_1 t} & 0\\ 0 & e^{\mu_2 t}\end{pmatrix}\mathrm{N}^{-1}\mathrm{A}_0\mathbf{x}_0$ which is in fact a linear combination of $e^{\mu_1 t}$ and $e^{\mu_2 t}$. That is

$$g'(t) = \sigma_1 e^{\mu_1 t} + \sigma_2 e^{\mu_2 t} \tag{3}$$

where $\sigma_{1,2} \in \mathbb{R}$. Obviously, $g(t) \to 0$ and $g'(t) \to 0$ as $t \to +\infty$. Note that σ_1 and σ_2 are not both zero. In fact, if σ_1 and σ_2 are both zero, i.e., $g'(t) \equiv 0$, then $\mathrm{N}\begin{pmatrix}e^{\mu_1 t} & 0\\ 0 & e^{\mu_2 t}\end{pmatrix}\mathrm{N}^{-1}\mathrm{A}_0\mathbf{x}_0 \in L$ for any $t \in \mathbb{R}$ which is contradictory to the fact that $\mathrm{N}\begin{pmatrix}e^{\mu_1 t} & 0\\ 0 & e^{\mu_2 t}\end{pmatrix}\mathrm{N}^{-1}\mathrm{A}_0\mathbf{x}_0 \to 0$as $t \to +\infty$.