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Transient flow in a linear reservoir for space–time fractional diffusion

C. Chen [a,*], R. Raghavan [b]

[a] Kappa Engineering, 11767 Katy Freeway, 500, Houston, TX 77079, USA
[b] Phillips Petroleum Co., (Ret.), P.O. Box 52756, Tulsa, OK 74152, USA

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ABSTRACT

A one-dimensional, fractional-order, transient diffusion equation is constructed to model diffusion in complex geological media. Such a conceptual model permits for the incorporation of a wide range of velocities as fluid particles in high and low permeability paths perform complex motions. The transient diffusion equation is non-local in character with both spatial and temporal fractional derivatives. The pressure distribution is derived in terms of the Laplace transformation and the Mittag–Leffler function. Results are used to deduce expectations in the early-time response of a fractured well producing complex reservoirs such as unconventional shales. The flux law considered here allows for declines in rate that are faster or slower than models based on classical diffusion. A brief survey of the Mittag–Leffler function and its computation is provided. We apply the results derived to obtain solutions for the 'trilinear' model that is often used to evaluate horizontal well performance.

1. Introduction

Transient diffusion in heterogeneous porous media is of long-standing interest in a number of disciplines. Conventional formulations assume that flux is directly proportional to the imposed pressure gradient. Understanding flow in media with complex geology and convoluted flow paths, however, requires formulation of flux laws that go beyond the requirement that the flux depends on the imposed gradient because fluid particles that trace complex paths no longer follow a space–time behavior based on Gauss's distribution that is a consequence of Darcy's law (Gefen et al., 1983; Le Mĕhautĕ and Crepy, 1983; Nigmatullin, 1984; Chang and Yortsos, 1990; Dassas and Duby, 1995; Caputo, 1998; Molz et al., 2002; Fomin et al., 2011a,b; Raghavan, 2011). The assumptions that underlie these proposals vary in scope from assuming that the diffusivity varies as a power law with distance, to assuming that the random motion of particles in the pores and fractures are a continuous time random walk (CTRW), or assuming that correlation scales are significantly large when compared with the scale of observation. Some of these formulations lead to fractional derivatives in time and space. But all lead to the conclusion that the mean square displacement (or mean square deviation) is not a Gaussian. The purpose of this paper is to examine the transient behavior in a linear reservoir where the flux law is non-local in that velocity distribution depends on both previous times and large regions in space by incorporating a spatial fractional derivative. The model discussed in this study may be used to evaluate well responses when a number of mechanisms that operate at different spatial and temporal scales govern the productive characteristics. This situation is typical of unconventional reservoirs produced subsequent to hydraulic fracturing. In this scheme the gradient in pressure at a point has by itself no physical interpretation in that it does not represent the flux. The expression we use is a generalization of the flux law used in Raghavan and Chen (2013a,b). By including the space domain, the structure of the solution changes in a significant way and expands the results in our earlier work as suggested in Raghavan (2011). A solution is obtained in terms of the Laplace transformation and sample results are documented through the use of the Stehfest algorithm (1970a,b). The results of this study have an immediate application to flow in unconventional reservoirs produced through hydraulic fractures. Anomalous diffusion will, in general, result in declines that are different from those predicated by conventional models and this is not unusual.

Fractional diffusion equations address the phenomena of long-range dependence and/or trapping events much better than models governing classical diffusion. Situations such as cracks, cervices and obstacles that slow down diffusion (subdiffusion) are modeled by time-fractional diffusion equations and situations where highly conductive, well connected paths that enhance diffusion (super-diffusion) are modeled by space-fractional diffusion equations. The need to model flow by fractional diffusion equations arises in a number of different contexts: assessing contaminant transport, evaluating diffusive behavior in fractured rocks modeled as fractals, and where mechanisms at different spatial and temporal scales govern production behavior. Applications of fractional derivatives in the context of contaminant transport have been discussed for a number of years (Lenormand, 1992; Benson et al., 2000) primarily because

* Corresponding author.
E-mail addresses: chen@kappaeng.com (C. Chen), Raghavan.Raj@gmail.com (R. Raghavan).

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Nomenclature

a	see (5.1) and Fig. 8
c	compressibility (L T^2/M)
$E_{\alpha,\beta}(z)$	Mittag–Leffler function; see (2.5) and (2.10)
$f(s)$	function defined in (5.1)
h	thickness (L)
k	permeability (L^2)
$k_{\alpha,\beta}$	see (2.1)
ℓ	reference length (L)
p	pressure (M/L/T^2)
p'	logarithmic derivative (M/L/T^2)
q	rate (L^3/T)
$q_D(x_D, t_D)$	dimensionless flux
t	time (T)
α	exponent
β	exponent
$\Gamma(x)$	gamma function
η	diffusivity; various
$\tilde{\eta}$	'diffusivity'; see (3.5) (L$^{\beta+1}$/T$^{\alpha}$)
λ	mobility (L T/M)
μ	viscosity (M/L/T)
ν	exponent; see (5.2)
ϕ	porosity (L^3/L^3)

Subscripts

D	dimensionless
i	coordinate, initial
w	well bore

Superscript

$-$	Laplace transform

plumes spread out faster than the square root of time predicted by Gaussian diffusion and also because plumes are often skewed rather than being symmetric (superdiffusion). Implicit in these studies is the use of the fractional form of Fick's law (Fomin et al., 2011a; Atangana and Kilicman, 2013). Others have examined contaminant transport in a multimodal context (Baeumer et al., 2001; Fomin et al., 2010, 2011a) where the contaminant partitions between mobile and immobile phases and where trapping is the dominant mechanism (subdiffusion). The influence of transport properties of the rock under subdiffusion through fractional derivatives was first considered in Park et al. (2000). This work models flow to a line-source and attempts to improve upon the fractal model of Chang and Yortsos (1990) through the fractional-time-derivative model proposed in Metzler et al. (1994). Over time, this work has been expanded by several authors; see, for example, Flamenco-Lopez and Camacho-Velazquez (2001), Camacho-Velázquez et al. (2008), Razminia et al. (2014), and Ali et al. (2014). Although not emphasized in many of these works, as noted in Raghavan (2011), use of these models requires the incorporation of either a fractional form of Darcy's law or a modification of the conservation equation. A fractional flux law is the better recourse. Situations that result in superdiffusion; that is, the role of extremely conductive paths is addressed in Cloot and Botha (2006). They modified, not surprisingly, the space terms of the classical diffusion equation in cylindrical coordinates. Here again, another fractional form of the flux law is required. The justification for the model chosen follows from their observation that the classical Theis (1935) solution is inadequate to model fractured rocks with highly conductive, well connected paths because observed responses are both under and over estimated at early and at later times, respectively. That long-range dependencies of the permeability profile are best addressed through the fractional space derivative is also noted in Herrick et al. (2002). The goal of our work, as already stated, will be to explore the combined effects of fractional space and time terms, and to examine early-time production responses at fractured wells producing fractured rocks where a number of mechanisms operate at different spatial and temporal scales.

Consideration of the model discussed here involves the evaluation of the Mittag–Leffler function in the Laplace domain; a non-trivial exercise (Verotta, 2010). Our code uses the algorithm described in Gorenflo et al. (2002); see also Berberan-Santos (2005) and Podlubny (2005). Often the computation of such functions is avoided. For our purposes we show that such a step is not advisable and outline our reasons. Based on experience our view is that, although not trivial, as already mentioned, the calculations are not formidable.

2. Some preliminary considerations

We briefly review the nonlocal flux law we use to derive the pressure distribution and briefly touch on the consequences of this law. The term nonlocal refers to the fact that in this work the flux at a point is influenced by factors besides the value of the pressure gradient at the point in question at a particular instant in time. Justification for such a law may be found in Molz et al. (2002), Cushman and Ginn (1993), Fomin et al. (2011a,b), Benson et al. (2013) among others. As we use the Laplace transformation to derive the solutions of interest, we arrive at expressions for the pressure distribution in terms of the Mittag–Leffler function. Accordingly, we briefly review this function and cover ground that is of direct relevance to this work.

2.1. The flux law

The expression for the flux law used here is of the form

$$v(x,t) = -\lambda_{\alpha,\beta}\frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}}\left[\frac{\partial^{\beta}}{\partial x^{\beta}}p(x,t)\right], \tag{2.1}$$

where $\lambda_{\alpha,\beta} = k_{\alpha,\beta}/\mu$. Both the exponents α and β are <1 with the temporal fractional derivative, $\partial^{\alpha}f(t)/\partial t^{\alpha}$, and spatial fractional derivative, $\partial^{\beta}f(x)/\partial x^{\beta}$, defined in the Caputo (1967) sense as

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t dt'(t-t')^{-\alpha}\frac{\partial}{\partial t'}f(t'), \tag{2.2}$$

and

$$\frac{\partial^{\beta}}{\partial x^{\beta}}f(x) = \frac{1}{\Gamma(1-\beta)}\int_0^x dx'(x-x')^{-\beta}\frac{\partial}{\partial x'}f(x'). \tag{2.3}$$

From (2.1) it is clear that the flux at a point, for example the production rate, will no longer be reflected by the instantaneous pressure gradient but to the values of pressure and gradients that extend over significant distances and also to previous times. The latter has been considered by us previously. Both phenomena that are characteristic of anomalous diffusion, namely long-range and trapping effects, may be considered through the flux law in (2.1). As we will see, the structure of the solution changes in a much more significant way with the incorporation of nonlocal spatial variables than with the incorporation of time dependent effects. It appears that the incorporation of α is akin to including local effects. The flux law in (2.1) will result in a mean square