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Gravitational waves from a Weyl-Integrable manifold: A new formalism

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ABSTRACT

We study the variational principle over an Hilbert–Einstein like action for an extended geometry taking into account torsion and non-metricity. By extending the semi-Riemannian geometry, we obtain an effective energy–momentum tensor which can be interpreted as physical sources. As an application we develop a new manner to obtain the gravitational wave equations on a Weyl-integrable manifold taking into account the non-metricity and non-trivial boundary conditions on the minimization of the action, which can be identified as possible sources for the cosmological constant and provides two different equations for gravitational waves. We examine gravitational waves in a pre-inflationary cosmological model.

1. Introduction

In the standard treatment to minimize the action, when a manifold has a boundary $\partial\mathcal{M}$, the action should be supplemented by a boundary term, so that the variational principle to be well-defined [1,2]. However, this is not the only manner to study this problem. As was recently demonstrated [3], there is another way to include the flux around a hypersurface that encloses a physical source without the inclusion of another term in the Hilbert–Einstein (HE) action. This treatment imposes a constraint on the dynamics obtained by varying the EH action. In that paper was demonstrated that the non-zero flux of the vector metric fluctuations through the closed 3D Gaussian-like hypersurface, is responsible for the gauge-invariance of gravitational waves. In present paper we are dealing with the variational principle over a Hilbert–Einstein like action, using an extended geometry with torsion and non-metricity, from which we obtain an effective energy–momentum tensor with sources in the torsion and the non-metricity. It can be viewed in a Riemannian geometry as a describing an effective stress tensor that represents a geometrically induced matter. Additionally, we develop a new manner to obtain gravitational waves on a Weyl-integrable manifold, which has non-metricity and nontrivial boundary terms included.

The paper is organized as follows: in the following section we shall study the general formalism. In Section 4 we examine the formalism in absence of torsion, taking into account purely Weylian contributions. In Section 5 we deal only with the contributions due to boundary terms. In Section 6 we examine an example in which massless gravitons are emitted during a pre-inflationary epoch of the universe. Finally, in Section 7 we develop some final remarks.

2. General formalism

We consider the variational principle in presence of torsion and non-metricity in an Hilbert–Einstein action. We shall start by considering an action in an extended geometry (i.e. a non-Riemannian manifold) conformed by a gravitational sector without the presence of matter in such generalized geometry. The Hilbert–Einstein action was extensively studied in Riemannian geometry [4], but we shall deal with an extended geometry:

$$\mathcal{S} = \frac{1}{2\kappa}\int_V d^4x\sqrt{-g}\,\mathcal{R}, \tag{1}$$

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where V denotes the volume of a spacetime manifold featured by a non-metricity [5,6] and a general torsion [5,7]. Furthermore, g is the determinant of the metric tensor $g_{\alpha\beta}$, the gravitational coupling is denoted by $\kappa = 8\pi G$ and $\mathcal{R} = g^{\alpha\beta}R_{\alpha\beta}$ is the scalar curvature. For a coordinate basis of the tangent space $\{\partial_\sigma\}$, the components of the Riemann tensor are given by

$$R^{\alpha}_{\beta\mu\nu} = \Gamma^{\alpha}_{\beta\nu,\mu} - \Gamma^{\alpha}_{\beta\mu,\nu} + \Gamma^{\sigma}_{\beta\nu}\Gamma^{\alpha}_{\sigma\mu} - \Gamma^{\sigma}_{\beta\mu}\Gamma^{\alpha}_{\sigma\nu}, \tag{2}$$

where the symbols $\Gamma^{\alpha}_{\mu\nu}$ denote the coordinate components for a generalized connection defined by [5,8]

$$\nabla_{\partial_\alpha}\partial_\beta = \Gamma^{\epsilon}_{\beta\alpha}\partial_\epsilon. \tag{3}$$

These components can be written in the general form

$$\Gamma^{\sigma}_{\mu\nu} = \left\{ {}^{\sigma}_{\mu\nu} \right\} + K^{\sigma}_{\mu\nu}, \tag{4}$$

where $\left\{ {}^{\gamma}_{\alpha\beta} \right\}$ are the components of the usual Riemannian connection (the second kind Christoffel symbols) and $K^{\sigma}_{\mu\nu}$ is a contortion tensor due to torsion and non-metricity, defined by [9]

$$K^{\sigma}_{\mu\nu} = -\frac{g^{\beta\sigma}}{2}\{\tau^{\alpha}_{\nu\beta}\,g_{\mu\alpha} + \tau^{\alpha}_{\mu\beta}\,g_{\alpha\nu} - \tau^{\alpha}_{\nu\mu}\,g_{\alpha\beta} + N_{\mu\beta\nu} + N_{\beta\nu\mu} - N_{\mu\nu\beta}\}, \tag{5}$$

such that $\tau^{\alpha}_{\mu\nu}$ and $N_{\alpha\beta\gamma}$ are respectively the torsion and the non-metricity tensors. For a coordinate basis, they are

$$\tau^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\nu\mu}, \tag{6}$$

$$N_{\alpha\beta\gamma} = g_{\beta\gamma;\alpha}, \tag{7}$$

where the semicolon denotes the covariant derivative defined in terms of the Γ connection (i.e. defined on the extended manifold).

Now, in order to derive the dynamical equations for gravitational waves on this general space-time manifold in a novel and consistent manner, we shall use a variational procedure. The variation of the action (1) leaves to the expression

$$\delta\left[\sqrt{-g}\,\mathcal{R}\right] = \sqrt{-g}\left[\delta g^{\alpha\beta}G_{\alpha\beta} + g^{\alpha\beta}\delta R_{\alpha\beta}\right], \tag{8}$$

where $G_{\alpha\beta} = R_{\alpha\beta} - (1/2)\mathcal{R}\,g_{\alpha\beta}$ is a generalization of the Einstein tensor, due to the fact that it is calculated in terms of the Γ connection (4). Therefore, it is easy to show that the generalized Einstein tensor contains contributions associated with both tensors, torsion and non-metricity. The last term between brackets of the Eq. (8) can be written as a generalized Palatini's identity in the form

$$g^{\alpha\beta}\,\delta R_{\alpha\beta} = W^{\mu}{}_{;\mu} - g^{\alpha\beta}{}_{;\mu}\,W^{\mu}_{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}(\delta\Gamma^{\mu}_{\sigma\beta}\,\tau^{\sigma}_{\alpha\mu} + \delta\Gamma^{\mu}_{\sigma\alpha}\,\tau^{\sigma}_{\beta\mu}), \tag{9}$$

where $W^{\mu} = g^{\alpha\beta}\,W^{\mu}_{\alpha\beta}$. Here, we have introduced the auxiliary tensor $W^{\mu}_{\alpha\beta}$ defined by

$$W^{\mu}_{\alpha\beta} = \delta\Gamma^{\mu}_{\alpha\beta} - \delta\Gamma^{\sigma}_{\sigma\beta}\,\delta^{\mu}_{\alpha}. \tag{10}$$

Inserting (9) in (8), and using the identity $g^{\alpha\nu}_{;\mu} = -g^{\beta\nu}\,g^{\alpha\sigma}\,N_{\sigma\beta\mu}$, we obtain the variation of the gravitational sector of the action (1)

$$\delta\mathcal{S} = \int_V d^4x\sqrt{-g}\,G_{\alpha\beta}\delta g^{\alpha\beta} + \int_V d^4x\sqrt{-g}\,W^{\mu}_{;\mu} + \int_V d^4x\sqrt{-g}\,N_{\mu\alpha\beta}W^{\beta\mu\alpha} - \frac{1}{2}\int_V d^4x\sqrt{-g}\,\zeta_{\alpha\beta}g^{\alpha\beta}, \tag{11}$$

where $\zeta_{\alpha\beta}$ is an auxiliary tensor field, given by

$$\zeta_{\alpha\beta} = \delta\Gamma^{\mu}_{\sigma\beta}\tau^{\sigma}_{\alpha\mu} + \delta\Gamma^{\mu}_{\sigma\alpha}\tau^{\sigma}_{\beta\mu}. \tag{12}$$

The third and fourth integrals in (11), are respectively related to the presence of non-metricity and torsion. On the other hand, the second integral in (11) can be reduced to a 3D hypersurface integral, in virtue of the Stokes Theorem:

$$\int_V d^4x\sqrt{-g}\,W^{\mu}_{;\mu} = \int_{\partial V} d^3x\sqrt{-g}\,W^{\mu}n_{\mu}, \tag{13}$$

where n_μ is a vector field which is normal to the hypersurface ∂V. It is usual in the literature to suppose that the surface integral must be neglected when the radius of ∂V is large enough to impose that the field $W^{\mu} \rightarrow 0$ in such limit, or when W^{μ} is tangent to ∂V, that is, when W^{μ} satisfies the relation $W^{\mu}n_{\mu} = 0$. In this paper we shall adopt a different path, and we shall see that the 3D hypersurface term is a source for the cosmological constant [3]. From the expression (13), can be noticed that the term $W^{\mu}_{;\nu} = g^{\alpha\beta}_{;\nu}\,W^{\mu}_{\alpha\beta} + g^{\alpha\beta}\,W^{\mu}_{\alpha\beta;\nu} \neq g^{\alpha\beta}\,W^{\mu}_{\alpha\beta;\nu}$ has contributions of non-metricity. This implies that, if we drop this term, the new contribution is not very important. However, when we neglect such boundary term, we must be careful with non-metricity.

We must notice that the Einstein tensor in (11) can be written as a Riemannian part, plus a non-Riemannian one, in the form

$$G_{\alpha\beta} = \bar{G}_{\alpha\beta} + K^{\mu}_{\mu\beta\,|\alpha} - K^{\mu}_{\alpha\beta\,|\mu} + K^{\nu}_{\mu\beta}K^{\mu}_{\nu\alpha} - K^{\nu}_{\alpha\beta}K^{\mu}_{\nu\mu} - \frac{g^{\sigma\gamma}}{2}(K^{\mu}_{\mu\gamma\,|\sigma} - K^{\mu}_{\sigma\gamma\,|\mu} + K^{\nu}_{\mu\gamma}K^{\mu}_{\nu\sigma} - K^{\nu}_{\sigma\gamma}K^{\mu}_{\nu\mu})\,g_{\alpha\beta}. \tag{14}$$

In this expression, the *bar* in $\bar{G}_{\alpha\beta}$ indicates that the Einstein tensor is calculated with the Levi-Civita connections. The symbol "|" denotes the Riemannian covariant derivative. It follows from (14) that when the non-metricity and the torsion vanish. The Einstein tensor in the first integral of (11), simply reduces to the usual Einstein tensor calculated with the Levi-Civita connections.

Now, using the Eqs. (4) and (5), the auxiliary tensor $W^{\sigma}_{\alpha\beta}$ and the vector field W^{σ}, can be written as

$$\begin{aligned}
W^{\mu}_{\alpha\beta} = \Bigg[&\frac{g^{\lambda\mu}}{2}\{\delta g_{\alpha\lambda,\beta} + \delta g_{\lambda\beta,\alpha} - \delta g_{\beta\alpha,\lambda} - \tau^{\rho}_{\beta\lambda}\,\delta g_{\alpha\rho} - \tau^{\rho}_{\alpha\lambda}\,\delta g_{\rho\beta}\} \\
&- \frac{\delta g^{\lambda\mu}}{2}\{g_{\alpha\lambda,\beta} + g_{\lambda\beta,\alpha} - g_{\beta\mu,\lambda} - \tau^{\rho}_{\beta\lambda}\,g_{\alpha\rho} - \tau^{\rho}_{\alpha\lambda}\,g_{\rho\beta} \\
&- N_{\alpha\lambda\beta} - N_{\lambda\beta\alpha} + N_{\beta\alpha\lambda}\} \\
&- \frac{g^{\lambda\sigma}}{4}(\delta g_{\lambda\sigma,\beta}\,\delta^{\mu}_{\alpha} + \delta g_{\lambda\sigma,\alpha}\,\delta^{\mu}_{\beta}) + \frac{\delta g^{\lambda\sigma}}{4} \\
&\times\ (g_{\lambda\sigma,\beta}\,\delta^{\mu}_{\alpha} + g_{\lambda\sigma,\alpha}\,\delta^{\mu}_{\beta} - N_{\lambda\sigma\beta}\,\delta^{\mu}_{\alpha} - N_{\lambda\sigma\alpha}\,\delta^{\mu}_{\beta})\Bigg] \\
&+ \Bigg[\frac{g^{\lambda\mu}}{2}\tau^{\rho}_{\beta\alpha}\,\delta g_{\rho\lambda} - \frac{g^{\lambda\mu}}{2}\tau^{\rho}_{\beta\alpha}\,g_{\rho\lambda} \\
&- \frac{g^{\lambda\sigma}}{4}(\delta g_{\lambda\sigma,\beta}\,\delta^{\mu}_{\alpha} - \delta g_{\lambda\sigma,\alpha}\,\delta^{\mu}_{\beta}) \\
&+ \frac{\delta g^{\lambda\sigma}}{4}(g_{\lambda\sigma,\beta}\,\delta^{\mu}_{\alpha} - g_{\lambda\sigma,\alpha}\,\delta^{\mu}_{\beta} + N_{\lambda\sigma\beta}\,\delta^{\mu}_{\alpha} - N_{\lambda\sigma\alpha}\,\delta^{\mu}_{\beta})\Bigg].
\end{aligned} \tag{15}$$

$$\begin{aligned}
W^{\mu} = g^{\alpha\beta}\Bigg[&\frac{g^{\lambda\mu}}{2}\Big\{\delta g_{\alpha\lambda,\beta} + \delta g_{\lambda\beta,\alpha} - \delta g_{\beta\alpha,\lambda} - \tau^{\rho}_{\beta\lambda}\,\delta g_{\alpha\rho} \\
&- \tau^{\rho}_{\alpha\lambda}\,\delta g_{\rho\beta}\Big\} - \frac{\delta g^{\lambda\mu}}{2}\Big\{g_{\alpha\lambda,\beta} + g_{\lambda\beta,\alpha} - g_{\beta\alpha,\lambda} \\
&- \tau^{\rho}_{\beta\lambda}\,g_{\alpha\rho} - \tau^{\rho}_{\alpha\lambda}\,g_{\rho\beta} \\
&- N_{\alpha\lambda\beta} - N_{\lambda\beta\alpha} + N_{\beta\alpha\lambda}\Big\} - \frac{g^{\lambda\sigma}}{4}\left(\delta g_{\lambda\sigma,\beta}\,\delta^{\mu}_{\alpha} + \delta g_{\lambda\sigma,\alpha}\,\delta^{\mu}_{\beta}\right) \\
&+ \frac{\delta g^{\lambda\sigma}}{4}\left(g_{\lambda\sigma,\beta}\,\delta^{\mu}_{\alpha} + g_{\lambda\sigma,\alpha}\,\delta^{\mu}_{\beta} - N_{\lambda\sigma\beta}\,\delta^{\mu}_{\alpha} - N_{\lambda\sigma\alpha}\,\delta^{\mu}_{\beta}\right)\Bigg].
\end{aligned} \tag{16}$$