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A mathematical model of fluctuation noise based on the wavelet transform

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Abstract

A new model of white noise based on the wavelet transform has been proposed. This model is more adequate for solving some radiophysical tasks, such as the problem of electromagnetic waves reflecting from the ionosphere. Moreover, it was shown that in terms of probabilistic description of the random-process trajectories, the wavelet implementation of this random process is more likely (using the probability density functional offered by Amiantov). The wavelet properties and the famous theorems of mathematical analysis and theory of chances were used to develop our model: the mean value theorem and Lyapunov's central limit theorem. Our study resulted in obtaining a theorem on random-process expansion in terms of wavelet basis. It was also shown that the obtained results were in agreement with those of Kotelnikov.

Keywords: White noise; Fluctuation noise; Wavelet transform.

According to Kotelnikov's works [1], normal fluctuation noise consists of a large number of pulses following each other in random time intervals; it is also possible for such pulses to overlap.

Ref. [1] and some other publications (see, for example, [2–4]), define the noise analytically as a trigonometric expansion. However, this model, characterized by a discrete spectrum, is not adequate for solving certain radiophysical problems. For example, in problems on radio-wave reflection in the lower (turbulent) ionosphere, it is more practical to use fluctuations with a continuous spectrum.

Let us take as a basis a probabilistic description of the possible trajectories of stationary random processes using the probability density functional $F(x(t))$, first introduced in Ref. [5]. In the case of a process that exhibits a flat power spectrum with a specific bandwidth, this functional is given by

$$F(x(t)) = h \exp\left(-\frac{1}{2N}\int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)dt\right), \qquad (1)$$

where $x(t)$ are the trajectories depending on the time t; h is the quantity depending on the partition rank of the interval on which the trajectories are examined (it is the same for all trajectories if the partition rank tends to zero); N is the height of the power spectrum; T is the time interval under consideration.

It is easy to see that under these conditions, when the functional is expressed by Eq. (1), the wavelet

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implementation of the process is more likely than the sine one. This follows from the fact that the integral in Eq. (1) diverges with an increase in T values for the sine function $x(t)$, while for the wavelet it is finite.

To further develop the results obtained in [1], our study has analytically defined the noise that is a superposition of elementary random processes as an expansion in terms of a wavelet basis. The proof of the theorem on the expansion of a stationary random process in terms of a wavelet basis is offered as a new result of our study.

For the purposes of further discussion, it is practical to use instead of the dimensional quantities t and T (that have a time dimension) the dimensionless ones, by dividing them, for example, to $t_0 = 1$ s; previous notations, namely t and T, will be used throughout the text for these quantities.

Theorem. *Let the W(t) noise be given in the interval [–T/2, T/2] by a superposition of 'elementary' random processes $F_k(t)$:*

$$W(t) = \sum_{k=1}^{N} F_k(t),$$

and the following conditions are fulfilled:

(1) *$F_k(t)$ are uncorrelated random processes;*
(2) *N, which is the number of pulses falling in the [–T/2, T/2] interval, is a random variable, and $N \gg 1$;*
(3) *the energy of the process*

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} W^2(t)dt$$

remains unchanged in each implementation of the random process.

Then from the standpoint of the mean-square convergence, the following expansion is valid:

$$W(t) = \frac{1}{\sqrt{T}C_\psi} \sum_i \sum_j 2^{-\frac{i}{2}} \left[\theta_{i,j} \sqrt{\sum_{k=1}^{N} M\left(\left(q_k^+\right)^2\right)} - \chi_{i,j} \sqrt{\sum_{k=1}^{N} M\left(\left(q_k^-\right)^2\right)} \right] \Psi_0\left(\frac{t-j}{2^i}\right), \quad (2)$$

where C_ψ is the normalization constant [4], equal to

$$C_\psi = \int_0^\infty \frac{\left|\hat{\Psi}_0(\omega)\right|^2}{\omega} d\omega$$

and depending on the chosen mother wavelet; the random variables $\Theta_{i,j}, \chi_{i,j} \epsilon N(0,1)$ q_k^+, q_k^- are, respectively, the areas of the positive and the negative ordinate sets of $F_k(t)$ implementations.

Proof. Let us use the commonly accepted notation

$$\Psi_{i,j}(t) = 2^{-\frac{i}{2}} \Psi_0\left(\frac{t-j}{2^i}\right)$$

for the wavelets [7] and examine wavelets $\Psi_{i,j}(t)$ and $\Psi_0(t)$ satisfying the following conditions when $T \to \infty$:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \Psi_0(t)dt \to 0, \quad (3)$$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \Psi_0(t)^2 dt \to 1. \quad (4)$$

Let us write the coefficients $W(i, j)$ of the expansion $W(t)$ in terms of these wavelets [6]:

$$W(i,j) = \sum_{k=1}^{N} \int_{-\frac{T}{2}}^{\frac{T}{2}} F_k(t)\Psi_{i,j}(t)dt.$$

The implementation of $F_k(t)$ is, generally speaking, an alternating function with an arbitrary number of zeroes (in its medium) that can be represented by tuples of positive and negative bursts, so that the expressions

$$W(i,j) = \sum_{k=1}^{N} C_k^+ - \sum_{k=1}^{N} C_k^-,$$

$$C_k^+ = \int_{-\frac{T}{2}}^{\frac{T}{2}} F_k^+(t)\Psi_{i,j}(t)dt,$$

$$C_k^- = \int_{-\frac{T}{2}}^{\frac{T}{2}} F_k^-(t)\Psi_{i,j}(t)dt \quad (5)$$

are true.

Let us consider the positive bursts and apply the mean value theorem known from mathematical analysis to the first integral of Eq. (5):

$$C_k^+ = \Psi_{i,j}(\bar{t}_k) q_k^+, \; q_k^+ = \int_{-\frac{T}{2}}^{\frac{T}{2}} F_k^+(t)dt, \; \bar{t}_k \in \left[-\frac{T}{2}; \frac{T}{2}\right]. \quad (6)$$

Let us evaluate the modulus of the covariance R of random quantities q_k^+ and $\Psi_{i,j}(\bar{t}_k)$ using the Cauchy–Bunyakovsky–Schwarz inequality:

$$R^2 \le D\left(\Psi_{i,j}(\bar{t}_k)\right) D\left(q_k^+\right),$$