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On burning a lump of coal

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ABSTRACT

Burning something, (*e.g.* the proverbial lump of coal, or an encyclopaedia for that matter), in a blackbody furnace leads to an approximately Planck emission spectrum with an average entropy/information transfer of approximately 3.9 ± 2.5 bits per emitted photon. This quantitative and qualitative result depends only on the underlying unitarity of the quantum physics of burning, combined with the statistical mechanics of blackbody radiation. The fact that the utterly standard and unitarity preserving process of burning something (in fact, burning anything) nevertheless *has* an associated entropy/information budget, and the quantitative *size* of that entropy/information budget, is a severely under-appreciated feature of standard quantum statistical physics.

1. Introduction

Burning a lump of coal in a furnace, or even burning an encyclopaedia, is (assuming the validity of standard quantum physics) generally agreed to be an exactly unitary process — with no associated information "puzzle". Nevertheless, there is a non-trivial entropy budget as (coarse graining) entropy is exchanged between the burning matter and the electromagnetic field, with a compensating quantity of information typically being "hidden" in photon–photon correlations.

Standard statistical mechanics reasoning applied to a furnace with a small hole, (or lamp-black surfaces for that matter), leads to the notion of blackbody radiation, with many basic features dating back to the 1840s. When combined with Planck's quantum hypothesis of 1900, one is quickly led to the notion of a Planck spectrum — with a prediction that any furnace with a small hole in one face will with high accuracy emit a Planck spectrum. Indeed commercially available blackbody furnaces designed along these lines provide a simple way of generating blackbody spectra commonly used for calibration purposes of all types.

(Suitable technical discussions may be found in references [1–4] and in the patent application [5]. The most up-to-date information is however only to be found on somewhat ephemeral commercial websites found by searching on the phrase "blackbody calibration furnace".)

While the underlying physical processes are manifestly unitary, implying strict conservation of the von Neumann entropy, $S_{\text{von Neumann}} = \text{tr}(\rho \ln \rho)$, the statistical mechanics reasoning that leads to the Planck spectrum inherently implies some coarse graining — one is agreeing to look only at *some* of the features of the photons that emerge from the hole in the face of the blackbody furnace, (the spectrum), and to not fixate on other features, (*e.g.*, the interstitial gaps), and also to ignore any photons that may remain in the furnace. That is, the coarse graining entropy depends very much on what exactly you *choose* to measure, (and what you *choose* to hide in the correlations with things you do not measure).

Under these circumstances, every photon that escapes the furnace has an energy $E = \hbar\omega$, and furthermore by definition the furnace has an associated temperature T. Thus every photon that escapes transfers a precisely quantifiable amount of thermodynamic entropy to the radiation field:

$$S = \frac{E}{T} = \frac{\hbar\omega}{T}. \tag{1}$$

After all, in transferring energy $E = \hbar\omega$ from the blackbody furnace to the radiation field at temperature T one is precisely satisfying the Clausius definition of entropy (and implicitly satisfying the Carathéodory definition of entropy); that the entropy in question ultimately depends on coarse graining (an agreement to not look behind the curtain) is not germane. We use this construction to *define* what we mean by the entropy of a single blackbody photon. We emphasise that this is not an *intrinsic* property of the photon; it is a *contingent* property based on knowing that the photon in question is coming from a blackbody furnace at the specified

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temperature. The main thrust of this article will be to quantify the entropy/information flows implicit in standard blackbody (Planckian spectrum) radiation in some detail.

2. Preliminaries

We start by noting that the concept of a blackbody furnace (blackbody cavity furnace) is utterly standard:

> A very good experimental approximation to a black body is provided by a cavity the interior walls of which are maintained at a uniform temperature and which communicates with the outside by means of a hole having a small diameter in comparison with the dimensions of the cavity. Any radiation entering the hole is partly absorbed and partly diffusely reflected a large number of times at the interior walls, only a negligible fraction eventually making its way out of the hole. — Zemansky [6].

Similar comments apply to any surface coated with "lamp black" (soot, carbon black). Based on these concepts, pre-quantum classical thermodynamics, (using Stefan's law and Stefan's constant σ, *aka* the Stefan–Boltzmann constant), quickly leads to a quantifiable notion of entropy density and energy density for isotropic black body radiation (implicitly assumed to be in internal equilibrium) [7–12]:

$$s = \frac{4}{3}\,\frac{4\sigma}{c}\,T^3; \quad \rho = \frac{4\sigma}{c}\,T^4; \quad s = \frac{4}{3}\,\frac{\rho}{T}. \tag{2}$$

Once one introduces quantum physics, this can be supplemented with a quantifiable notion of photon number density [9,10,12]. For the Bose energy distribution relevant to the Planck spectrum the number (per unit volume) of photons in the frequency range $(\omega, \omega + \mathrm{d}\omega)$ is:

$$dn = f(\omega)\,\mathrm{d}\omega = \frac{1}{\pi^2 c^3}\,\frac{\omega^2 \mathrm{d}\omega}{e^{\hbar\omega/k_B T} - 1}. \tag{3}$$

This just depends on Bose statistics and phase space.[1] The photon number density (for isotropic blackbody radiation) can be written as [9,10,12]

$$n = \frac{2\zeta(3)}{\pi^2}\left(\frac{k_B T}{\hbar c}\right)^3, \tag{4}$$

and the entropy density (for isotropic blackbody radiation) can be rewritten as [9,10,12]

$$s = \frac{4\pi^2}{45}\left(\frac{k_B T}{\hbar c}\right)^3 k_B. \tag{5}$$

The introduction of quantum physics has allowed us to *derive* Stefan's constant σ in terms of the more primitive physical constants $\hbar$, k_B, and c. Consequently, (for an isotropic photon gas of blackbody radiation implicitly assumed in internal equilibrium), the entropy per photon can be seen to be[2]

$$S_{\text{per photon}} = \frac{s}{n} = \frac{2\pi^4}{45\,\zeta(3)}\,k_B. \tag{6}$$

This will slightly differ from our results below, by a purely kinematic factor of 4/3, simply because, (instead of dealing with an isotropic photon gas in internal equilibrium), we shall be more interested in individual photons being exchanged between the blackbody furnace and the wider environment. We include this present version of the argument because it can easily be tracked back all the way to quite standard textbook material. In the more subtle version of the argument presented below we shall also consider various moments in the distribution, not just the average.

Now consider the effect of coarse graining the entropy. If we start from some initial primitive notion of von Neumann entropy, (which is conserved under unitary evolution), then coarse graining leads to:

$$S_{\text{coarse grained}} = S_{\text{before coarse graining}} + S_{\text{correlations}}. \tag{7}$$

We can also rephrase this in terms of information, (negentropy; negative entropy [13,14]), as follows:

$$\begin{aligned} S_{\text{before coarse graining}} &= S_{\text{coarse grained}} - S_{\text{correlations}} \\ &= S_{\text{coarse grained}} + I_{\text{correlations}}. \end{aligned} \tag{8}$$

Focussing on our single-photon definition of entropy, it is often convenient to measure entropy in "natural units" ("nats", sometimes called "nits" or "nepits"), constructed by dividing by the Boltzmann constant [15,16]. This leads to a dimensionless notion of entropy:

$$\hat{S} = \frac{S}{k_B} = \frac{E}{k_B T} = \frac{\hbar\omega}{k_B T}. \tag{9}$$

It is often convenient to further convert entropy to an equivalent number of bits [13,14,17–19], (sometimes rephrased in terms of "Shannons" with symbol Sh [20]), by using the Boltzmann formula, (relating entropy to the number of microstates), to write

$$S = k_B \ln\Omega = k_B \ln(2^N) = N k_B \ln 2, \tag{10}$$

which thereby justifies the definition

$$\hat{S}_2 = \frac{S}{k_B \ln 2} = \frac{\hat{S}}{\ln 2} = \frac{\hbar\omega}{k_B T \ln 2}. \tag{11}$$

For the purposes of this article we will always be evaluating dimensionless entropies, either in terms of "nats" (*i.e.* $\hat{S}$) or in terms of bits (*i.e.* $\hat{S}_2$).

3. Entropy/information in blackbody radiation

Using the Bose distribution the *average energy* per photon in blackbody radiation is given by the standard result

$$\langle E\rangle = \hbar\,\langle\omega\rangle = \hbar\,\frac{\int \omega f(\omega)\mathrm{d}\omega}{\int f(\omega)\mathrm{d}\omega} = \frac{\pi^4}{30\,\zeta(3)}\,k_B T. \tag{12}$$

Consequently, the *average entropy* per photon in blackbody radiation is simply

$$\begin{aligned} \langle\hat{S}\rangle &= \frac{\langle E\rangle}{k_B T} = \frac{\hbar\langle\omega\rangle}{k_B T} = \frac{\pi^4}{30\,\zeta(3)} \\ &\approx 2.701178034 \text{ nats/photon}. \end{aligned} \tag{13}$$

This implies

$$\begin{aligned} \langle\hat{S}_2\rangle &= \frac{\pi^4}{30\,\zeta(3)\ln 2} \\ &\approx 3.896976153 \text{ bits/photon}. \end{aligned} \tag{14}$$

This is purely a blackbody statistical mechanics result. Note this result certainly applies to burning a lump of coal in a furnace,

[1] Fixing the absolute normalization (though straightforward) is often not really needed as it will drop out of many calculations.

[2] Note this is a completely flat-space result, gravity simply does not have any relevance for the present computation.