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Entanglement entropy of scattering particles

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ABSTRACT

We study the entanglement entropy between the two outgoing particles in an elastic scattering process. It is formulated within an S-matrix formalism using the partial wave expansion of two-body states, which plays a significant role in our computation. As a result, we obtain a novel formula that expresses the entanglement entropy in a high energy scattering by the use of physical observables, namely the elastic and total cross sections and a physical bound on the impact parameter range, related to the elastic differential cross-section.

1. Introduction

Entanglement is a significant concept which appears in various subjects of quantum physics. The quantum entanglement has been attracting much attention of theoretical physicists, since remarkable progress in the entanglement between the systems on two regions was made in quantum field theories [1] and holography [2], and the intriguing conjecture called ER = EPR [3] was suggested. In the context of the ER = EPR conjecture, the entanglements between two particles, which are, for example, a pair of accelerating quark and anti-quark [4] and a pair of scattering gluons [5], have been studied. Then it naturally induces the following primitive question: How does the entanglement entropy of a pair of particles change from an initial state to a final one in an elastic channel of scattering process? It is qualitatively expected that the elastic collision of two initial particles, *e.g.*, in a high energy collider, generates some amount of entanglement between the particles in the final state. We are interested in quantifying the entanglement entropy generated by collision.

By just neglecting inelastic channels in weak coupling perturbation [6], Ref. [7] analyzed such entanglement entropy in a field theory by the use of an S-matrix.[1] In this article we exploit the S-matrix formalism further in order for a non-perturbative understanding of the entanglement entropy in a scattering process with also an inelastic channel to be taken into account. This is especially required in the case of strong interaction scattering at high energy where inelastic multi-particle scattering contributes to a large part of the total cross-section, while elastic scattering is still important. The basic S-matrix formalism of strong interaction, as developed long time ago, *e.g.*, in Refs. [9,10], allows us to find an approach to scattering processes without referring explicitly to an underlying quantum field theory.

Following Refs. [9,10], we consider a scattering process of two incident particles, A and B, whose masses are m_A and m_B respectively, in $1+3$ dimensions. This process is divided [9] into the following two channels:

$$\text{“Elastic” channel:} \quad \mathrm{A}+\mathrm{B} \to \mathrm{A}+\mathrm{B}$$

$$\text{“Inelastic” channel:} \quad \mathrm{A}+\mathrm{B} \to \mathrm{X}$$

where X stands for any possible states except for the two-particle state, A + B. We postpone the study extended to a matrix including more varieties of two-particle channels [10] to a further publication.

The full Hilbert space of states is not usually factorized as $\mathcal{H}_{\text{full}} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_X$ in an interacting system. However the Hilbert space of both the initial and final states is factorizable in the S-matrix formalism, because one considers only asymptotic initial and final states long before and after the interaction. We introduce the S-matrix, $\mathcal{S}$, for the overall set of initial and final states. Once we fix an initial state $|\text{ini}\rangle$, the final state $|\text{fin}\rangle$ is determined by the S-matrix. In this article we are interested in the entanglement between two outgoing particles, A + B, in a final

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[1] We quote for completion Ref. [8], where the entanglement entropy is discussed in a low energy decay process using different concept and method.

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state of elastic scattering in the presence of a non-negligible fraction of open inelastic final states. Therefore we additionally introduce a projection operator Q onto the two-particle Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ from $\mathcal{H}_{\rm full}$. Then the final elastic state, in other words, the state of two outgoing particles, is described as $|{\rm fin}\rangle = Q\mathcal{S}|{\rm ini}\rangle$.

We employ the two-particle Fock space $\{|\vec{p}\,\rangle_A\} \otimes \{|\vec{q}\,\rangle_B\}$ as the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. The two-particle state which consists of particle A with momentum $\vec{p}$ and B with $\vec{q}$ is denoted by $|\vec{p},\vec{q}\,\rangle = |\vec{p}\,\rangle_A \otimes |\vec{q}\,\rangle_B$. We define an inner product of the two-particle states in a conventional manner by $\langle \vec{p},\vec{q}\,|\vec{k},\vec{l}\,\rangle = 2E_{A\vec{p}}\delta^{(3)}(\vec{p}-\vec{k})\,2E_{B\vec{q}}\delta^{(3)}(\vec{q}-\vec{l}\,)$, where $E_{I\vec{p}} = \sqrt{p^2+m_I^2}$ $(I=A,B)$ and $p=|\vec{p}|$.

We shall study the entanglement between the two outgoing particles, A and B. When the density matrix of the final state on $\mathcal{H}_A \otimes \mathcal{H}_B$ is denoted by ρ, we define a reduced density matrix as $\rho_A = {\rm tr}_B\,\rho$. Then the entanglement entropy is given by $S_{\rm EE} = -\,{\rm tr}_A\,\rho_A \ln \rho_A$. The other way to calculate the entanglement entropy is to use the Rényi entropy, $S_{\rm RE}(n) = (1-n)^{-1}\ln {\rm tr}_A(\rho_A)^n$. It leads to the entanglement entropy described as $S_{\rm EE} = \lim_{n\to 1} S_{\rm RE}(n) = -\lim_{n\to 1}\frac{\partial}{\partial n}{\rm tr}_A(\rho_A)^n$.

2. Partial wave expansion

The partial wave expansion is often useful to analyze a scattering process. Before starting to study the entanglement entropy, let us recall what Refs. [9,10] studied.

We adopt a center-of-mass frame. The state of the two particles, A + B, which have momenta $\vec{p}$ and $-\vec{p}$, is denoted by $|\vec{p}\,\rangle\!\rangle := |\vec{p},-\vec{p}\,\rangle$, while the many-particle state of X is denoted by $|X\rangle$. Since the complete set of states is given by the orthogonal basis, $\{|\vec{p}\,\rangle\!\rangle, |X\rangle\}$, one can describe the identity matrix as

$$\mathbf{1} = \int \frac{d^3\vec{p}}{2E_{A\vec{p}}2E_{B\vec{p}}\delta^{(3)}(0)}|\vec{p}\,\rangle\!\rangle\langle\!\langle\vec{p}| + \int dX\,|X\rangle\langle X|\,. \tag{2.1}$$

We notice that $\delta^{(3)}(0)$ comes from $\langle\!\langle\vec{k}|\vec{l}\,\rangle\!\rangle = 2E_{A\vec{k}}2E_{B\vec{k}}\delta^{(3)}(\vec{k}-\vec{l}\,)\delta^{(3)}(0)$, due to our definition of the inner product of states.

One can expand the S-matrix elements in term of partial waves. Let us consider the S-matrix and T-matrix defined by $\mathcal{S} = \mathbf{1} + 2i\mathcal{T}$. The unitarity condition is $\mathcal{S}^\dagger\mathcal{S} = \mathbf{1}$, which is equivalent to $i(\mathcal{T}^\dagger - \mathcal{T}) = 2\mathcal{T}^\dagger\mathcal{T}$. Extracting the factor of energy–momentum conservation, we describe the T-matrix elements as

$$\begin{aligned}
&\langle\!\langle\vec{p}|\mathcal{T}|\vec{q}\,\rangle\!\rangle = \delta^{(4)}(P_{\vec{p}} - P_{\vec{q}})\langle\!\langle\vec{p}|\mathbf{t}|\vec{q}\,\rangle\!\rangle\,,\\
&\langle\!\langle\vec{p}|\mathcal{T}|X\rangle = \delta^{(4)}(P_{\vec{p}} - P_X)\langle\!\langle\vec{p}|\mathbf{t}|X\rangle\,.
\end{aligned} \tag{2.2}$$

$P_{\vec{p}}$ and P_X are the total energy-momenta of $|\vec{p}\,\rangle\!\rangle$ and $|X\rangle$ respectively, which say $P_{\vec{p}} = (E_{A\vec{p}} + E_{B\vec{p}}, 0, 0, 0)$.

One introduces the overlap matrix $F_{\vec{p}\,\vec{k}}(k,\cos\theta)$,

$$F_{\vec{p}\,\vec{k}} = \frac{2\pi k}{E_{A\vec{k}} + E_{B\vec{k}}}\int dX\langle\!\langle\vec{p}|\mathbf{t}^\dagger|X\rangle\delta^{(4)}(P_X - P_{\vec{k}})\langle X|\mathbf{t}|\vec{k}\rangle\!\rangle\,, \tag{2.3}$$

where k and θ are defined by $\vec{p}\cdot\vec{k} = pk\cos\theta$ and $k=p$. This matrix implies the contribution of the inelastic channel at the middle of the scattering process. The T-matrix element in the elastic channel and the overlap matrix are decomposed in terms of partial waves,

$$\frac{\pi k}{E_{A\vec{k}} + E_{B\vec{k}}}\langle\!\langle\vec{p}|\mathbf{t}|\vec{k}\rangle\!\rangle = \sum_{\ell=0}^{\infty}(2\ell+1)\tau_\ell(k)P_\ell(\cos\theta)\,, \tag{2.4}$$

$$F_{\vec{p}\,\vec{k}}(k,\cos\theta) = \sum_{\ell=0}^{\infty}(2\ell+1)f_\ell(k)P_\ell(\cos\theta)\,, \tag{2.5}$$

where $P_\ell(\cos\theta)$ are the Legendre polynomials. Then one can rewrite the unitarity condition as

$${\rm Im}\,\tau_\ell = |\tau_\ell|^2 + \frac{f_\ell}{2}\,. \tag{2.6}$$

Using $s_\ell := 1 + 2i\tau_\ell$, which comes from the partial wave expansion of the S-matrix element,

$$\frac{\pi k}{E_{A\vec{k}} + E_{B\vec{k}}}\langle\!\langle\vec{p}|\mathbf{s}|\vec{k}\rangle\!\rangle = \sum_{\ell=0}^{\infty}(2\ell+1)s_\ell P_\ell(\cos\theta)\,, \tag{2.7}$$

the unitarity condition is equivalent to $s_\ell^* s_\ell = 1 - 2f_\ell$. If there is not an inelastic channel, *i.e.* $f_\ell = 0$, then the unitarity condition is reduced to $s_\ell^* s_\ell = 1$. A comment in order [9,10] is that we can define a pseudo-unitary two-body S-matrix with partial wave components, $\omega_\ell^*\omega_\ell = 1$, by rescaling s_ℓ as $\omega_\ell := s_\ell/\sqrt{1-2f_\ell}$.

The partial wave expansion allows us to depict the integrated elastic cross section, the integrated inelastic cross section and the total cross section as

$$\begin{aligned}
&\sigma_{\rm el} = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)|\tau_\ell|^2\,, \quad \sigma_{\rm inel} = \frac{2\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)f_\ell\,,\\
&\sigma_{\rm tot} = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\,{\rm Im}\,\tau_\ell\,.
\end{aligned} \tag{2.8}$$

The differential elastic cross section is

$$\begin{aligned}
\frac{d\sigma_{\rm el}}{dt} &= \frac{\pi}{k^4}\sum_{\ell,\ell'}(2\ell+1)(2\ell'+1)\tau_\ell\tau_{\ell'}^* P_\ell(\cos\theta)P_{\ell'}(\cos\theta)\\
&= \frac{|A|^2}{64\pi s k^2}\,,
\end{aligned} \tag{2.9}$$

where $A(s,t)$ is the scattering amplitude, s and t are the Mandelstam variables, and the scattering angle $\cos\theta = 1 + t/(2k^2)$.

3. Entanglement entropy of two particles

We consider two unentangled particles, A and B, with momenta $\vec{k}$ and $\vec{l}$ as incident particles. That is to say, we choose a single state as an initial state;

$$|{\rm ini}\rangle = |\vec{k},\vec{l}\,\rangle = |\vec{k}\rangle_A \otimes |\vec{l}\,\rangle_B\,. \tag{3.1}$$

Here we have not taken the center-of-mass frame yet. Of course the entanglement entropy of the initial state vanishes. In terms of the S-matrix, the final state of two particles, $|{\rm fin}\rangle = Q\mathcal{S}|{\rm ini}\rangle$, is described as

$$|{\rm fin}\rangle = \left(\int \frac{d^3\vec{p}}{2E_{A\vec{p}}}\frac{d^3\vec{q}}{2E_{B\vec{q}}}|\vec{p}\,,\vec{q}\,\rangle\langle\vec{p}\,,\vec{q}\,|\right)\mathcal{S}|\vec{k},\vec{l}\,\rangle\,. \tag{3.2}$$

Then we can define the total density matrix of the final state by $\rho := \mathcal{N}^{-1}|{\rm fin}\rangle\langle{\rm fin}|$. The normalization factor $\mathcal{N}$ will be determined later so that ρ satisfies ${\rm tr}_A\,{\rm tr}_B\,\rho = 1$. Tracing out ρ with respect to the Hilbert space of particle B, we obtain the reduced density matrix, $\rho_A := {\rm tr}_B\,\rho$, namely,

$$\begin{aligned}
\rho_A = \frac{1}{\mathcal{N}}\int &\frac{d^3\vec{p}}{2E_{A\vec{p}}}\frac{d^3\vec{q}}{2E_{B\vec{q}}}\frac{d^3\vec{p}\,'}{2E_{A\vec{p}'}}\\
&\times\left(\langle\vec{p}\,,\vec{q}\,|\mathcal{S}|\vec{k},\vec{l}\,\rangle\langle\vec{k},\vec{l}\,|\mathcal{S}^\dagger|\vec{p}\,',\vec{q}\,\rangle\right)|\vec{p}\,\rangle_{A\,A}\langle\vec{p}\,'|\,.
\end{aligned} \tag{3.3}$$

Now let us adopt the center-of-mass frame, which leads to $\vec{k}+\vec{l} = 0$. Then the initial state is $|{\rm ini}\rangle = |\vec{k}\rangle\!\rangle$, and the reduced density matrix becomes

$$\rho_A = \frac{1}{\mathcal{N}}\int \frac{d^3\vec{p}}{2E_{A\vec{p}}}\frac{\delta(0)\delta(p-k)}{4k(E_{A\vec{k}} + E_{B\vec{k}})}\left|\langle\!\langle\vec{p}\,|\mathbf{s}|\vec{k}\rangle\!\rangle\right|^2|\vec{p}\,\rangle_{A\,A}\langle\vec{p}\,|\,, \tag{3.4}$$