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A duality web of linear quivers

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ABSTRACT

We show that applying the Bailey lemma to elliptic hypergeometric integrals on the A_n root system leads to a large web of dualities for $\mathcal{N}=1$ supersymmetric linear quiver theories. The superconformal index of Seiberg's SQCD with $SU(N_c)$ gauge group and $SU(N_f)\times SU(N_f)\times U(1)$ flavour symmetry is equal to that of N_f-N_c-1 distinct linear quivers. Seiberg duality further enlarges this web by adding new quivers. In particular, both interacting electric and magnetic theories with arbitrary N_c and N_f can be constructed by quivering an s-confining theory with $N_f=N_c+1$.

Supersymmetric gauge theories are a highly active subject of study and many discoveries were made in this field in the past decades. One particularly interesting phenomenon is duality: for certain strongly coupled supersymmetric quantum field theories, there exist weakly coupled dual theories that describe the same physical system in terms of different degrees of freedom. A famous example is Seiberg duality [1] for $\mathcal{N}=1$ supersymmetric quantum chromodynamics (SQCD), where two dual theories, referred to as electric and magnetic, flow to the same infrared (IR) theory. While such dualities are hard to prove, supersymmetric theories allow for the definition of observables that are independent of the description, i.e. they should yield the same result on both sides of the duality. One such quantity is the superconformal index (SCI) [2,3], which counts the number of BPS states of a given theory. It turns out that SCIs are related to elliptic hypergeometric functions, which have also found many other applications in physics.

A long hunt for the most general possible exactly solvable model of quantum mechanics has led to the discovery of elliptic hypergeometric integrals forming a new class of transcendental special functions [4]. In the first physical setting these integrals served either as a normalization condition of particular eigenfunctions or as eigenfunctions of the Hamiltonian of an integrable Calogero–Sutherland type model [5]. The Bailey lemma for such integrals [6] appeared to define the star-triangle relation associated with quantum spin chains [7]. However, a major physical application was found by Dolan and Osborn [8] who showed that certain elliptic hypergeometric integrals are identical to SCIs of $4d$ supersymmetric field theories and that Seiberg duality can be understood in terms of symmetries of such integrals. In [9], many explicit examples were studied. In the present work, we describe a web of dualities that can be constructed using the Bailey lemma of [6] and [10]. Starting from a known elliptic beta integral on the A_n root system [11] that is identified with the star-triangle relation, one gets an algorithm for constructing an infinite chain of symmetry transformations for elliptic hypergeometric integrals. The emerging integrals can be interpreted as the SCIs of linear quiver gauge theories, a possibility that was already mentioned in [9].

Quiver gauge theories are theories with product gauge groups that arise as world volume theories of branes placed on singular spaces or from brane intersections [12–14]. Their field content can be depictured by so-called quiver diagrams; all new theories discussed in this article are of this type. Note that while the quivers we discuss are also linear like those described in [15], field content and flavour symmetries are different.

This letter is dedicated to applying an integral extension of the standard Bailey chains techniques [16] to SCIs. We identify the star-triangle relation (a variant of the Yang–Baxter equation) with an elliptic hypergeometric integral on the A_n root system that corresponds to the superconformal index of an s-confining $\mathcal{N}=1$ $SU(N_c)$ gauge theory. The main result of our calculation is that the SCI of SQCD with $SU(N_c)$ gauge group and $SU(N_f)\times SU(N_f)\times U(1)$ flavour symmetry is equal to that of N_f-N_c-1 distinct linear quivers. Seiberg duality leads to magnetic partners for these quivers, some of which are again dual to yet other quivers. In total, this leads to a very large duality web, composed of Seiberg and Bailey lemma dualities. An example of such a web corresponding to the electric SQCD with $N_c=3$ and $N_f=6$ is illustrated in

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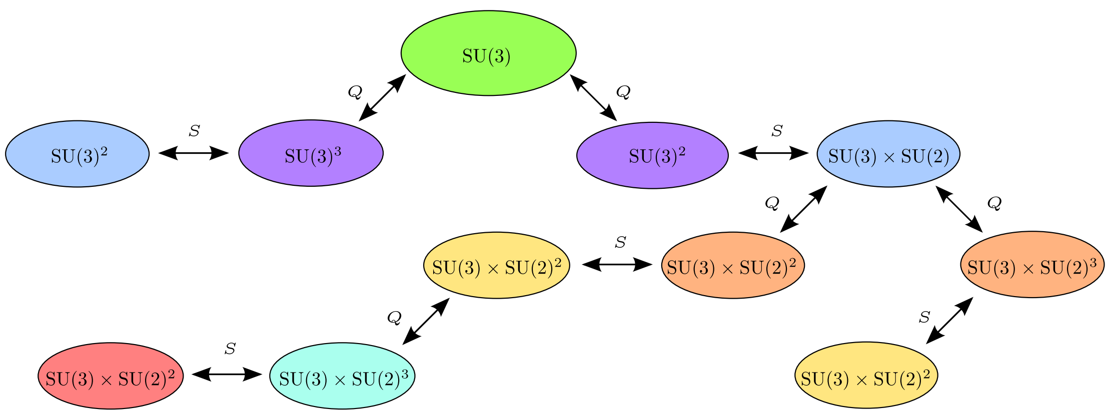

Fig. 1. The duality web corresponding to the electric part of SQCD for $N_c = 3$ and $N_f = 6$. Q denotes a duality obtained from Eq. (11) and S denotes Seiberg duality of Eq. (6). In total, there are ten distinct quiver gauge theories dual to the original theory.

Fig. 1. Another nontrivial consequence is that indices of both electric and magnetic interacting theories can be constructed from a simple s-confining theory.

The SCI of $\mathcal{N} = 1$ theories is defined as

$$\mathcal{I} = \mathrm{Tr}(-1)^{\mathcal{F}} e^{-\beta H} p^{\frac{R}{2}+J_R+J_L} q^{\frac{R}{2}+J_R-J_L} \prod_i z_i^{G_i} \prod_j y_j^{F_j}, \tag{1}$$

where $\mathcal{F}$ is the fermion number, R is the R-charge, J_L and J_R are the Cartan generators of the rotation group $SU(2)_L \times SU(2)_R$, and G_i and F_j are maximal torus generators of the gauge and flavour groups. The theory is assumed to be compactified on a spatial three-sphere, hence the name “sphere index”. As shown in [17] (see also [18] and [19]), in this case the SCI is proportional to the partition function of the theory, where p and q are variables of the three-sphere metric and the parameters y_j are interpreted as mean values of the background gauge fields of the flavour group. The index only receives contributions from states with $H = E - 2J_L - \frac{3}{2}R = 0$, E being the energy, and is independent of the chemical potential β. In order to obtain a gauge invariant expression, an integral over the gauge group is performed, which gives the explicit expression

$$\mathcal{I}(p,q,y) = \int_G d\mu(g) \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} i(p^n, q^n, y^n, z^n)\right), \tag{2}$$

where $d\mu(g)$ is the group measure and the function $i(p,q,y,z)$ denotes the single-particle state index. The latter is determined by representation theory through

$$i(p,q,y,z) = \frac{2pq - p - q}{(1-p)(1-q)} \chi_{adj}(z) + \sum_j \frac{(pq)^{\frac{r_j}{2}} \chi_j(y)\chi_j(z) - (pq)^{\frac{1-r_j}{2}} \overline{\chi}_j(y)\overline{\chi}_j(z)}{(1-p)(1-q)}, \tag{3}$$

where r_j are R-charges, $\chi_{adj}(z)$ is the character of the adjoint representation under which the gauge fields transform, while the second term is a sum over the chiral matter superfields that contains the characters of the corresponding representations of the gauge and flavour groups. In the following, we make use of the fact that SCIs are identical to particular elliptic hypergeometric integrals.

Define the generalized A_n-elliptic hypergeometric integral as

$$I_n^{(m)}(\mathbf{s},\mathbf{t}) = \kappa_n \int_{\mathbb{T}^n} \frac{\prod_{j=1}^{n+1}\prod_{l=1}^{n+m+2} \Gamma(s_l z_j^{-1}, t_l z_j)}{\prod_{1\le j<k\le n+1} \Gamma(z_j z_k^{-1}, z_j^{-1} z_k)} \prod_{k=1}^{n} \frac{dz_k}{2\pi i z_k}, \tag{4}$$

with $\prod_{j=1}^{n+1} z_j = 1$, $\kappa_n = (p;p)^n (q;q)^n/(n+1)!$, $\mathbf{s} = (s_1, \ldots, s_{n+m+2})$, $\mathbf{t} = (t_1, \ldots, t_{n+m+2})$, $|s_i|, |t_i| < 1$ and the balancing condition $\prod_{i=1}^{n+m+2} s_i t_i = (pq)^{m+1}$. The q-Pochhammer symbol is defined as $(z;q)_\infty = \prod_{k=0}^{\infty}(1 - zq^k)$, and the elliptic gamma function as

$$\Gamma(z) := \Gamma(z;p,q) = \prod_{j,k=0}^{\infty} \frac{1 - z^{-1} p^{j+1} q^{k+1}}{1 - z p^j q^k}, \tag{5}$$

$$\Gamma(a,b) := \Gamma(a;p,q)\Gamma(b;p,q),$$

for $z \in \mathbb{C}^*$ and $|p|, |q| < 1$. Eq. (4) can be interpreted as the SCI of an $\mathcal{N} = 1$ theory with gauge group $SU(N_c)$ for $N_c = n+1$ and a vector multiplet in its adjoint representation. There is a chiral multiplet in the fundamental and one in the antifundamental of the gauge group, each transforming in the fundamental representation of one of the factors of the flavour group $SU(N_f) \times SU(N_f)$, for $N_f = n + m + 2$. Furthermore, there is a global $U(1)_V$ symmetry and the R-symmetry $U(1)_R$. Note that for the sake of brevity, we will not list any R-charges in this paper, as they can be easily recovered from the integral expressions. As shown in [8], Seiberg duality is realized by the general integral identity [20]

$$I_n^{(m)}(\mathbf{s},\mathbf{t}) = \prod_{j,k=1}^{n+m+2} \Gamma(t_j s_k) I_m^{(n)}(\mathbf{s}', \mathbf{t}') \tag{6}$$

with the arguments $\mathbf{s}' = (S^{\frac{1}{m+1}}/s_1, \ldots, S^{\frac{1}{m+1}}/s_{n+m+2})$ and $\mathbf{t}' = (T^{\frac{1}{m+1}}/t_1, \ldots, T^{\frac{1}{m+1}}/t_{n+m+2})$, where $S = \prod_{j=1}^{n+m+2} s_j$, $T = \prod_{j=1}^{n+m+2} t_j$, $ST = (pq)^{m+1}$ and $|t_k|, |s_k|, |S^{\frac{1}{m+1}}/s_k|, |T^{\frac{1}{m+1}}/t_k| < 1$. The operation $n \leftrightarrow m$ gives the correct dual symmetry groups since $N_f = n + m + 2 \to N_f$ and $N_c = n + 1 \to m + 1 = N_f - N_c$.

For $m = 0$, Eq. (6) reduces to the exact evaluation formula [4,11]

$$I_n^{(0)}(\mathbf{s},\mathbf{t}) = \prod_{k=1}^{n+2} \Gamma\left(\frac{S}{s_k}, \frac{T}{t_k}\right) \prod_{k,l=1}^{n+2} \Gamma(s_k t_l). \tag{7}$$

This is an example of s-confinement [21]: the infrared is described only by gauge-invariant operators, and the origin of the classical moduli space remains a vacuum even after quantizing the theory (chiral symmetry is intact). Furthermore, a confining superpotential is generated dynamically.

We define [6,10] as a Bailey pair with respect to the parameter t a pair of functions $\alpha(z,t)$ and $\beta(w,t)$ satisfying the relation $\beta(w,t) = M(t)_{wz}\alpha(z,t)$, where $M(t)_{wz}$ is an elliptic hypergeometric integral operator. The (integral) Bailey lemma states that given such a pair of functions, one automatically obtains another