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Unified theory in the worldline approach

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ABSTRACT

We explore unified field theories based on the gauge groups $SU(5)$ and $SO(10)$ using the worldline approach for chiral fermions with a Wilson loop coupling to a background gauge field. Representing path ordering and chiral projection operators with functional integrals has previously reproduced the sum over the chiralities and representations of standard model particles in a compact way. This paper shows that for $SU(5)$ the $\bar{\mathbf{5}}$ and $\mathbf{10}$ representations – into which the Georgi–Glashow model places the left-handed fermionic content of the standard model – appear naturally and with the familiar chirality. We carry out the same analysis for flipped $SU(5)$ and uncover a link to $SO(10)$ unified theory. We pursue this by exploring the $SO(10)$ theory in the same framework, the less established unified theory based on $SU(6)$ and briefly consider the Pati–Salam model using $SU(4) \times SU(2) \times SU(2)$.

1. Introduction

The worldline formalism [1,2] is a first quantised approach to field theory and offers a powerful alternative tool for theoretical calculations. Quantities in the field theory are re-expressed as one-dimensional quantum mechanical transition amplitudes of spinning point particles. In this context, a recent model of chiral fermions demonstrated an interesting way of summing over the gauge group representations and chiralities present in the standard model [3]. This sum was constructed for a single generation of fermions supplemented by a sterile neutrino. The model is substantially different from the usual field theory approach because the assignment of particles to their group representations and chiralities arises naturally, rather than being pre-determined by hand. The model also has a computational simplicity compared to more traditional methods in field-theory which require the evaluation of a complicated sum over these representations. Instead that sum is generated through the evaluation of a single functional determinant. In this letter we will generalise that result by considering a variety of other symmetry groups that are familiar from previous studies into grand unified theories.

Progress in the worldline description of chiral particles is central to a formulation of the standard model in first quantised language, where the worldline formalism can offer significant computational advantages over calculations in perturbative quantum field theory [4,5]. Furthermore the first quantised model presented in [3] has an underlying string theory [6] which generalises to non-Abelian interactions so it is natural to consider the consequences of using different symmetry groups in that context.

The motivation for considering alternative gauge groups is the unification of the electroweak and strong interactions. The purpose of this unification is to find a theory with only one coupling constant, from which the standard model emerges after spontaneous symmetry breaking as a low-energy effective theory [7]. The gauge group with smallest rank that can accommodate the standard model is $SU(5)$. This is the famous Georgi–Glashow model [8]. We shall demonstrate that the representations and chiralities of the standard model particles as described by the standard $SU(5)$ and flipped $SU(5)$ unified theories can also be generated with the new approach of [3].

The main results we shall arrive at for the representations and chiralities of standard model particles will be found to agree with well-known results in the literature. They can be arrived at by a variety of other group theoretic methods but we believe that the relative compactness of the new approach, combined with the fact that particle multiplets are no longer arbitrarily chosen, means that this approach has some merit as a complementary tool to more conventional methods.

This letter is laid out as follows. The next section briefly reviews the argument and notation in [3] and in Section 3 the model is applied to the unified theories of $SU(5)$ and flipped $SU(5)$. We also consider other unified theories which appear in the literature, namely $SU(6)$, $SO(10)$ and $SU(4) \times SU(2) \times SU(2)$.

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2. Fields and worldlines

We consider a left- or right-handed massless fermion moving in a background gauge field, A. We take A to transform in the adjoint representation of some symmetry group which is described by anti-Hermitian Lie algebra generators $\{T_S\}$. Working in Euclidean space, the action for a left-handed massless fermion field, ξ, is

$$S\left[\bar{\xi},\xi\right]=\int d^4x\, i\xi^{\dagger}\bar{\sigma}\cdot D\xi \tag{1}$$

where $D=(\partial+A)$ and $\sigma^{\mu}=\left(\mathbb{1},\sigma^{i}\right)$ make up the Euclidean Dirac operator $\bar{\sigma}\cdot D$ (the coupling strength is absorbed into A). Following the worldline approach requires us to functionally integrate over the matter field to arrive at the effective action $\Gamma[A]$. In this case, however, we must avoid the well-known problem of how to define the determinant of the Dirac operator acting on chiral fermions transforming in a non-real representation of the gauge group. We can, however, define the phase-difference of determinants which motivates us to consider the variation of the effective action under an infinitesimal change in A [9,10]. This is easily found to be

$$\begin{aligned}\delta_A\Gamma[A]&=\delta_A\ln\int\mathscr{D}\left(\bar{\xi},\xi\right)e^{-S\left[\bar{\xi},\xi\right]}\\&=\mathrm{Tr}\left(\left(\bar{\sigma}\cdot D\right)^{-1}\bar{\sigma}\cdot\delta A\right)\end{aligned}\tag{2}$$

which can be written in terms of γ-matrices[1] as

$$-\int_0^{\infty}dT\,\mathrm{Tr}\left(\frac{(1-\gamma_5)}{2}e^{T(\gamma\cdot D)^2}\gamma\cdot D\,\gamma\cdot\delta A\right).\tag{3}$$

We recognise in (3) the heat kernel of the operator $(\gamma\cdot D)^2=D^2\mathbb{1}+\frac{1}{2}\gamma^{\mu}F_{\mu\nu}\gamma^{\nu}$ and in [3] a worldline representation of this expression was derived:

$$\begin{aligned}\delta_A\Gamma[A]=&-\int_0^{\infty}\frac{dT}{T}\oint_{L/R}\mathscr{D}\omega\mathscr{D}\psi\,e^{-S[w,\psi]}\\&\times\mathscr{P}\,\mathrm{tr}\left(g(2\pi)\int_0^{2\pi}dt\,\psi\cdot\dot{\omega}\,\psi\cdot\delta A\right).\end{aligned}\tag{4}$$

Here $\omega^{\mu}(t)$ describes a point particle traversing a closed loop (which generates the functional trace) and the Grassmann variables ψ^{μ} are the spin degrees of freedom living on that worldline. The action $S[w,\psi]$ is just a gauge fixed version of Brink, Di Vecchia and Howe's description [11] of the dynamics of a spin 1/2 point particle:

$$S[\omega,\psi]=\frac{1}{2}\int_0^{2\pi}\frac{\dot{\omega}^2}{T}+\psi\cdot\dot{\psi}\,dt\,,\tag{5}$$

from which the integration measure $\frac{dT}{T}$ in (4) can be understood as the Faddeev–Popov determinant associated with the fixing of a local worldline supersymmetry.

Upon quantisation the fundamental anti-commutation relations $\left\{\psi^{\mu},\psi^{\nu}\right\}=\delta^{\mu\nu}$ can be solved by taking $\psi^{\mu}=\frac{1}{\sqrt{2}}\gamma^{\mu}$ which shows that the role of the ψ^{μ} is to represent the γ-matrices. The coupling of the fermion to the gauge field is provided by $g(t)$ – this is the super-Wilson loop which is familiar from quantum field theory and is often encountered in the worldline approach [1]:

$$g(t)=\mathscr{P}\exp\left(-\int_0^{t}\mathcal{A}^S(t)T_S\,dt\right)\tag{6}$$

where

$$\mathcal{A}=\dot{\omega}\cdot A+\frac{T}{2}\psi^{\mu}F_{\mu\nu}\psi^{\nu}.\tag{7}$$

The L/R subscript in (4) denotes the boundary conditions on ψ which are interpreted depending on the chirality of the fermion. For left-handed fermions the path integral with periodic boundary conditions on ψ is subtracted from that with anti-periodic boundary conditions whereas for right-handed fermions the two contributions are summed. These combinations insert the appropriate projection operators $1\mp\gamma^5$ into the path integral. For a field theory describing a number of different particles, such as the standard model, one would also need to form the sum of (4) over the representations and chiralities of the full matter content. This summation needs to be implemented manually and is determined by the theorist's choice of the assignment of particles into their multiplets.

The path ordering prescription in (6) is required in a non-Abelian theory to ensure gauge invariance of the coupling to the gauge field but it complicates the evaluation of the functional integrals. The conventional way to deal with the non-Abelian nature of the coupling is to perturbatively expand the effective action and to impose the path ordering by hand [2,12,13]. However, there are other approaches to dealing with the non-commutative character of the Wilson-loop exponent such as by the introduction of additional Grassmann fields [14]. This was the approach taken in [3] which we now review.

The path ordering can be represented with functional integrals by introducing a set of anti-commuting operators $\tilde{\phi}_r$ and ϕ_s satisfying $\{\tilde{\phi}_r,\phi_s\}=\delta_{rs}$ with action $S_{\phi}=\int\tilde{\phi}\cdot\dot{\phi}\,dt$ [15–17]. It is easy to check the following definition furnishes us with a representation of the Lie algebra

$$R^S\equiv\tilde{\phi}_rT^S_{rs}\phi_s;\qquad\left[R^S,R^T\right]=if^{STU}R^U,\tag{8}$$

which can be used to absorb the gauge group indices in the Wilson-loop exponent. So instead of working directly with (4) we will find it advantageous to combine the above ideas to consider as it stands the related quantity

$$\int_0^{\infty}\frac{dT}{T}\int\mathscr{D}\omega\mathscr{D}\psi\,e^{-S[w,\psi]}\int_0^{2\pi}dt\,\psi\cdot\dot{\omega}\,\psi\cdot\delta A\,\frac{\delta Z[\mathcal{A}]}{\delta\mathcal{A}}\tag{9}$$

where

$$Z[\mathcal{A}]=\int\mathscr{D}\tilde{\phi}\mathscr{D}\phi\,e^{-\int_0^{2\pi}\tilde{\phi}\left(\frac{d}{dt}+\mathcal{A}\right)\phi}\tag{10}$$

is responsible for producing the interaction between the fermion and the gauge field.

This theory has been studied using worldline techniques before [18,19], where the focus has been on its canonical quantisation. In particular, the Fock space built by acting on the vacuum with anti-commuting creation operators can be described by wave function components which transform as anti-symmetric tensor products of the representation of the gauge group generators. Acting on wave functions of the form $\Psi(x,\tilde{\phi})$ the creation and annihilation operators can be represented by $\phi^{\dagger}=\tilde{\phi}$ and $\phi=\partial_{\tilde{\phi}}$. Then the wave functions have a finite Taylor expansion

$$\begin{aligned}&\Psi(x,\tilde{\phi})=\\&\Psi(x)+\tilde{\phi}_{r_1}\Psi^{r_1}(x)+\tilde{\phi}_{r_1}\tilde{\phi}_{r_2}\Psi^{[r_1r_2]}(x)+\ldots+\tilde{\phi}_{r_1}\tilde{\phi}_{r_2}..\tilde{\phi}_{r_N}\Psi^{[r_1r_2..r_N]}(x)\end{aligned}\tag{11}$$

[1] We use $\gamma^0=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ and $\gamma^j=\begin{pmatrix}0&i\sigma^j\\i\bar{\sigma}^j&0\end{pmatrix}$.