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Lie transformation method on quantum state evolution of a general time-dependent driven and damped parametric oscillator

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ABSTRACT

A variety of dynamics in nature and society can be approximately treated as a driven and damped parametric oscillator. An intensive investigation of this time-dependent model from an algebraic point of view provides a consistent method to resolve the classical dynamics and the quantum evolution in order to understand the time-dependent phenomena that occur not only in the macroscopic classical scale for the synchronized behaviors but also in the microscopic quantum scale for a coherent state evolution. By using a Floquet U-transformation on a general time-dependent quadratic Hamiltonian, we exactly solve the dynamic behaviors of a driven and damped parametric oscillator to obtain the optimal solutions by means of invariant parameters of Ks to combine with Lewis–Riesenfeld invariant method. This approach can discriminate the external dynamics from the internal evolution of a wave packet by producing independent parametric equations that dramatically facilitate the parametric control on the quantum state evolution in a dissipative system. In order to show the advantages of this method, several time-dependent models proposed in the quantum control field are analyzed in detail.

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1. Introduction

The classical dynamics of a driven and damped harmonic oscillator (DDHO) is a fundamental problem discussed in many textbooks and all its behaviors are well known [1,2]. However, this simple model can be used to understand the dynamics of more general systems embedded in different dissipative environments and driven by arbitrary external forces. A more complicated system possesses a richer frequency structure of internal dynamics but only one frequency window plays the dominant role in a certain parametric region. The external driving force often maintains an energy input to stimulate or to control the dynamics of the system. A more complex driving force often induces similar dynamics as that does by a simple periodic driving due to the limited bandwidth of a responsive window. Up to now, some behaviors of the damped harmonic oscillator driven by a simple periodic force still exhibit exciting dynamics, especially due to the rapidly developing fields of the optomechanical systems [3] and the quantum shortcut control problem [4]. In this paper, we want to reconsider this model in a unified framework from both classical and quantum mechanical points of view to exactly solve the dynamical equation for the sake of dynamic controls based on a general time-dependent model. In many control problems, DDHO is the simplest but fundamental model to reveal the main properties of a controlled system. Although the dynamics of a classically forced and damped harmonic oscillator will finally follow the external driving force and its behavior is completely deterministic, its transient behavior to a final state or its dynamical response to an external force is still an interesting topic to be explored in the quantum regime from a control perspective. For any classical or quantum control problems, the controlled systems are definitely nonconservative and the corresponding theories for an arbitrary time-dependent Hamiltonian, which should exhibit rich and novel behaviors beyond the perturbation theory and the adiabatic theory, are still under development [5]. In this paper, we will extensively investigate a general time-dependent model, a driven and damped parametric oscillator (DDPO), in a unified view of Lie transformation method to obtain not only the classical and quantum dynamics, but also the parametric connections to the Lewis–Riesenfeld invariant method. In order to provide a complete description of this model, we first give a brief review on the classical dynamics of DDHO to lay down a basic knowledge for the classical motion of DDPO, and then completely solve the problem in the quantum regime by using a time-dependent transformation method based on Lie algebra.

2. Classical dynamics of DDHO

In order to identify dynamic differences between classical and quantum behaviors, we first briefly review the main properties of the classical dynamics of a DDHO. The equation of motion to govern a classical DDHO with constant parameters is ($m = 1$)

$$\ddot{x}(t) + 2\gamma\dot{x}(t) + \omega_0^2 x(t) = F(t)\,, \tag{1}$$

where ω_0 is the internal frequency, γ is the damping rate related to the Q factor by $Q = \omega_0/2\gamma$, and $F(t)$ is the external driving force. Eq. (1) is a fundamental equation to understand a general damped and driven oscillators emerging in many physical systems. Besides the traditional weak vibration in a mechanical system, one important model in the plasma, called Lorentz oscillator, is used to describe the motion of a charged particle (trapped ion) driven by the electromagnetic fields. The other typical model is about the electronic current in the RLC circuits or the electromagnetic field in a finesse cavity pumping by an input field. Surely, many other phenomena in chemical, biological or economic fields can also be explained or understood by DDHO model [1,2].

The dynamics of a damped oscillator under a driving force exhibits a very important phenomenon: resonance, and many behaviors in this world can be explained or controlled by the resonant effect. As most of the driving or control signals can be decomposed into harmonic components, most studies of DDHO focus on a harmonic driving force and Eq. (1) simplifies to

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = F_0\cos(\Omega t + \phi)\,, \tag{2}$$

where F_0, Ω and ϕ are the driving strength, frequency and initial phase, respectively. Eq. (2) can be analytically solved and Fig. 1(a) demonstrates two typical motions of the oscillator driven by