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Vortex structure of Gross–Pitaevskii model

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Abstract

The vortex line of the Gross–Pitaevskii model is studied. The kinetic helicity of the vortex is discussed, and vortex structure is classified by the Hopf index, linking number in geometry. A mechanism of generation and annihilation of vortex lines is given by the method of phase singularity theory. The dynamic behavior of the vortex at the critical points is discussed in detail, and three kinds of length approximation relations at the neighborhood of a critical point are given: $l \propto (t - t^*)^{1/2}$, $l \propto t - t^*$, $l = \text{const}$.

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1. Introduction

The dramatic achievement of Bose–Einstein condensation at ultralow temperatures in experiments [1,2] on vapors of rubidium and sodium has stimulated an intense interest in the production of vortices and theoretical investigations of their structure, energy, dynamics, and stability [3,4]. The condensates of alkali vapours are pure and dilute, so that the Gross–Pitaevskii (GP) model which represents the so-called mean-field limit of quantum field theories gives a precise description of the atomic condensates and their dynamics at low temperatures. This situation differs from that for superfluid helium [5], where the

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relatively high density and strong repulsive interactions greatly complicate the analytical treatments, so that the GP mode provides at most a qualitative description.

Knotted vortices have been studied widely in different physical situations such as hydrodynamics [6,7], field theory [8], non-linear excited media [9], and optics [10] since Lord Kelvin's knotted vortex atom hypothesis [11]. In this paper, the knotted vortex lines of the GP model are studied by the method of phase singularity theory. Dirac recognized important role of phase singularity of quantum mechanical wave function in his work on monopoles [12], and Madelung gave a vivid interpretation of the lines where the phase is singular [13] on the hydrodynamic formulation of the Schrödinger theory. These are the vortex lines in the flow of the probability fluid.

We know that a single component BEC can be described by a single particle wave function of N bosons of mass m. The wave function obeys the Gross–Pitaevskii equation [14,15]

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + W_0\psi||\psi||^2. \tag{1}$$

Here W_0 characterizes the potential between bosons, assumed positive in our treatment.

The velocity field $\mathbf{V}$ is defined in terms of the probability current

$$\mathbf{V} = \frac{\mathrm{i}\hbar}{2m\psi^*\psi}(\psi\nabla\psi^* - \psi^*\nabla\psi). \tag{2}$$

The wave function is usually written as

$$\psi = ||\psi||\,\mathrm{e}^{\mathrm{i}\varphi}, \tag{3}$$

where φ is a phase factor. The velocity is just gradient of the phase factor, i.e.,

$$\mathbf{V} = \frac{\hbar}{m}\bigtriangledown\varphi. \tag{4}$$

It leads to a trivial curl-free result

$$\bigtriangledown\times\mathbf{V} = 0. \tag{5}$$

Therefore, the flow is strictly irrotational in the bulk. Feynman [16] found that this statement has to be modified. He pointed out that the curl of the velocity can be non-zero at a singular line, the core of quantum vortex line. So vorticity may live only on the lines of singularities of the phase. Based on the phase singularity theory, we not only obtain the correct result of curl of velocity field, but also get the precise expression of kinetic helicity of vortex lines.

This paper is organized as follows. In the second section, we have classified the topological structure of vortex lines of GP model in terms of Hopf index and Brouwer degree; In the third section, we have studied the topological structure of knotted vortex lines, given the relation of helicity with linking number in geometry. In the fourth section, we have given a mechanism of generation and annihilation of vortex lines.

2. Classification of vortex lines

We denote the condensate wave function as

$$\psi = \psi^1 + \mathrm{i}\psi^2. \tag{6}$$