Physics Letters A 379 (2015) 1555–1562

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Towards an exact relativistic theory of Earth's geoid undulation

Sergei M. Kopeikin [a,c,*], Elena M. Mazurova [b,c], Alexander P. Karpik [c]

[a] Department of Physics & Astronomy, University of Missouri, Columbia, MO 65211, USA
[b] Moscow State University of Geodesy and Cartography, 4 Gorokhovsky Alley, Moscow 105064, Russia
[c] Siberian State Geodetic Academy, 10 Plakhotny St., Novosibirsk 630108, Russia

ARTICLE INFO

Article history:
Received 14 November 2014
Received in revised form 19 February 2015
Accepted 23 February 2015
Available online 11 April 2015
Communicated by P.R. Holland

Keywords:
Gravity
Relativity
Geodesy
Geoid
Undulation

ABSTRACT

The present paper extends the Newtonian concept of the geoid in classic geodesy towards the realm of general relativity by utilizing the covariant geometric methods of the perturbation theory of curved manifolds. It yields a covariant definition of the anomalous (disturbing) gravity potential and formulates differential equation for it in the form of a covariant Laplace equation. The paper also derives the Bruns equation for calculation of geoid's height with full account for relativistic effects beyond the Newtonian approximation. A brief discussion of the relativistic Bruns formula is provided.

1. Introduction

Knowledge of the figure and size of the Earth is vitally important in geophysics and in applied sciences for determining precise position of objects on Earth's surface and in near space, depicting correctly topographic maps, creating digital terrain models, and many others. Solution of this problem is challenging for the real figure of the Earth has an irregular shape which can be neither described by a simple analytic expression nor easily computed as mass distribution of the Earth is not known well enough [1]. To manage solution of this problem, C.F. Gauss proposed to take one of the equipotential surfaces of Earth's gravitational field as a mathematical idealization approximating the real shape of the Earth such that it coincides with the mean sea level of idealized oceans representing the surface of homogeneous water masses at rest, subject only to the force of gravity and free from variations with time [2]. In 1873, a German mathematician J.B. Listing[1] coined the term *geoid* to describe this mathematical surface and, since then, the geoid has become a subject of a considerable scientific investigation in geodesy, oceanography, geophysics, and other Earth sciences [3]. Geoid's equipotential surface is perpendicular everywhere to the gravity force vector defining direction of the plumb line. In its own turn, the direction of plumb line is defined by the law of distribution of mass density inside Earth's crust and mantle. For the mass distribution is basically uneven, the shape of geoid's surface is not an ellipsoid of revolution with regularly varying curvature.

The Stokes–Poincaré theorem has played a major role in developing the theory of Earth's figure: if a body of total mass M rotates with constant angular velocity Ω about a fixed axis, and if $\mathcal{S}$ is a level surface of its gravity field enclosing the entire mass, then the gravity potential in the exterior space of $\mathcal{S}$ is uniquely determined[2] by M, Ω, and the parameters defining $\mathcal{S}$ [2]. However, geodesy is more interested in the inverse problem of the theory of Earth's figure which is to determine the shape of geoid from observed values of gravitational field.

Geoid's precise calculation is usually carried out by combining a global geopotential model of gravitational field with terrestrial gravity anomalies measured in the region of interest and supplemented with the local/regional topographic information. The gravity anomalies (along with other modern methods [2]) allow us to find out the undulation of geoid's surface that is measured with respect to a reference level surface of the World Geodetic System [4] established in 1984 (WGS84), and last revised in 2004. This reference surface is called a reference ellip-

* Corresponding author at: Department of Physics & Astronomy, University of Missouri, Columbia, MO 65211, USA.
E-mail addresses: kopeikins@missouri.edu (S.M. Kopeikin), e_mazurova@mail.ru (E.M. Mazurova), rector@ssga.ru (A.P. Karpik).

[1] It is the same J.B. Listing who introduced in 1847 the term *topology* in mathematics.

[2] In classic geodesy Earth's angular velocity is denoted ω. However, this symbol is commonly used in general relativity to denote vorticity, and we employ it later on in relativistic equations.

http://dx.doi.org/10.1016/j.physleta.2015.02.046
0375-9601/

soid. Geoid's undulation is given in terms of height above the ellipsoid taken along the normal line to ellipsoid's surface (see http://earth-info.nga.mil/GandG/wgs84/ for more detail).

A reference level surface, $\bar{\mathcal{S}}$, is defined by the condition of constant gravity potential, $\bar{U}_{\rm N}$, generated by a perfect fluid being rigidly rotated with respect to celestial reference frame [5] with a constant angular velocity Ω,

$$\bar{U}_{\rm N}(r,\theta) \equiv \bar{V}(r,\theta) + \frac{1}{2}\Omega^2 r^2 \sin^2\theta\,, \tag{1}$$

where $x^i = \{x^1, x^2, x^3\} = \{r, \theta, \lambda\}$ are spherical coordinates: r – radius-vector, θ – the polar angle (co-latitude) measured from the rotational axis, and λ – the longitude measured in the equatorial plane. Eq. (1) also defines the surfaces of constant density and pressure of the fluid [2].

The quantity $\bar{V} = \bar{V}(r,\theta)$ in (1) is the axisymmetric gravitational potential determined inside the mass distribution by the Poisson equation,

$$\Delta_{\rm N}\bar{V}(r,\theta) = -4\pi G\bar{\rho}\,, \tag{2}$$

where $\bar{\rho} = \bar{\rho}(r,\theta)$ is the axisymmetric volume mass density, G is the Newtonian gravitational constant,

$$\Delta_{\rm N} \equiv \partial_{rr} + \frac{2}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta} + \frac{1}{r^2\tan\theta}\partial_\theta + \frac{1}{r^2\sin^2\theta}\partial_{\lambda\lambda}\,, \tag{3}$$

is the Laplace operator in spherical coordinates, and the partial derivatives $\partial_i \equiv \partial/\partial x^i$, $\partial_{ij} \equiv \partial^2/\partial x^i \partial x^j$ (the Roman indices take values $1, 2, 3$). Inside masses a differential equation defining gravity potential, $\bar{U}_{\rm N}$, is

$$\Delta_{\rm N}\bar{U}_{\rm N} = -4\pi G\bar{\rho} + 2\Omega^2\,, \tag{4}$$

and is mostly used in geophysics.

Physical geodesy uses the Laplace equation

$$\Delta_{\rm N}\bar{V}(r,\theta) = 0\,, \tag{5}$$

instead of (2) as the gravitational field is only required outside masses for all relevant applications. Laplace's equation (5) is fully sufficient to determine the gravitational potential V in the exterior space, where the density distribution has not to be known. Nonetheless, it is worth emphasizing that solution of the Laplace equation (5) is not fully arbitrary but must match with a solution of the Poisson equation (2) with physically meaningful mass density distribution inside Earth's body.

Because all functions depend only on r and θ, the reference surface is an axisymmetric body. In the most general case, Eq. (1) does not define a surface of the ellipsoid of revolution. Only in case of a uniform mass density, $\bar{\rho} = $ const., the reference level surface coincides with the ellipsoid of revolution [6, Section 5.2]. The homogeneous ellipsoid of revolution is very convenient as a reference surface because its external (called *normal*) gravity field can be modeled by closed formulas. In principle, it is possible to construct level spheroids that provide a better fit to the geoid but their equations are more complicated mathematically and do not significantly reduce deviation of geoid from ellipsoid. Hence, they are less suitable as physical normal figures [2, Section 4.2.1].

When applying general relativity to calculation of geoid's surface, it becomes important to realize that the post-Newtonian reference level surface cannot be the ellipsoid of revolution any longer. The reason is that a figure of reference in geodesy is to be a solution of the Newtonian gravity field equation (4). The same principle must be hold in general relativity. It requires to find out an exact interior solution of the Einstein gravity field equations that would be consistent with the solution representing the homogeneous ellipsoid of revolution in classic geodesy. This general-relativistic problem is not trivial from mathematical point of view, because of non-linearity of Einstein's equations, and has not yet been solved. Calculations conducted in the post-Newtonian approximations reveal that the uniformly rotating perfect fluid with homogeneous density is not an ellipsoid but represents an axisymmetric surface of a higher polynomial order [7–10] but the convergence of the post-Newtonian series has not yet been explored. In this situation, the only restriction which we impose in the present paper on the shape of the reference level surface is that it is consistent with the Einstein equations.

Earth's crust is a thin surface layer having irregular mass density that deviates significantly from the axisymmetric distribution. Furthermore, the Earth mantle shows a non-axisymmetric surface deformation which easily reaches the same dimension as the crust variation, and its density is much bigger than the density of the crust. Because of these irregularities in both crust and mantle, the physical surface, $\mathcal{S}$, of the geoid is perturbed and deviates from the equipotential surface $\bar{\mathcal{S}}$ of the unperturbed (axisymmetric) figure defined by (1). We introduce the overall mass density perturbation of both the mantle and the crust by equation

$$\mu(r,\theta,\lambda) \equiv \rho(r,\theta,\lambda) - \bar{\rho}(r,\theta)\,, \tag{6}$$

where $\rho(r,\theta,\lambda)$ is the actual density of Earth's matter. We denote the actual gravity potential of Earth by

$$W_{\rm N}(r,\theta,\lambda) \equiv V(r,\theta,\lambda) + \frac{1}{2}\Omega^2 r^2 \sin^2\theta\,, \tag{7}$$

where $V = V(r,\theta,\lambda)$ is a gravitational potential that is determined by the Poisson equation

$$\Delta_{\rm N}V(r,\theta,\lambda) = -4\pi G\rho(r,\theta,\lambda)\,, \tag{8}$$

inside masses, and the Laplace equation

$$\Delta_{\rm N}V(r,\theta,\lambda) = 0\,, \tag{9}$$

outside masses.

We call the difference

$$T_{\rm N}(r,\theta,\lambda) \equiv W_{\rm N}(r,\theta,\lambda) - \bar{U}_{\rm N}(r,\theta)\,, \tag{10}$$

the disturbing (Newtonian) potential where both functionals, $W_{\rm N}$ and $\bar{U}_{\rm N}$, are calculated at the same point of space under assumption that the angular velocity Ω remains unperturbed. It is straightforward to see that the disturbing potential obeys to

$$\Delta_{\rm N}T_{\rm N}(r,\theta,\lambda) = -4\pi G\mu(r,\theta,\lambda)\,, \tag{11}$$

inside mass distribution, and to the Laplace equation

$$\Delta_{\rm N}T_{\rm N}(r,\theta,\lambda) = 0\,, \tag{12}$$

outside masses.

Molodensky [11,12] reformulated (12) into an integral equation

$$2\pi T_{\rm N} + \oiint_{\Sigma} \frac{T_{\rm N}}{\ell} n^i \partial_i \ln(\ell T_{\rm N})\, d\Sigma = 0\,, \tag{13}$$

where $\ell = |\boldsymbol{x} - \boldsymbol{x}'|$ denotes the distance between the source point, $\boldsymbol{x}'$, taken on Earth's surface Σ and the field point, $\boldsymbol{x}$, while $d\Sigma$ is the surface element of integration at point $\boldsymbol{x}'$, and n^i is the (outward) unit normal to Σ at $\boldsymbol{x}'$. The physical surface Σ of the Earth is known from the Global Navigation Satellite System (GNSS) measurements [1]. Thus, the only remaining unknown in (13) is the disturbing potential, $T_{\rm N}$. It can be found from (13) by employing the gravity disturbances of $T_{\rm N}(\Sigma)$ taken on Σ as boundary values [13]. As soon as $T_{\rm N}$ is known everywhere in space, the geoid's undulation (its height $\mathfrak{N}$ above the reference ellipsoid) can be found from Bruns' equation [1]