Results in Physics 6 (2016) 252–255

Contents lists available at ScienceDirect

Results in Physics

journal homepage: www.journals.elsevier.com/results-in-physics

Time evolution equations for hydrodynamic variables with arbitrary initial data

A. Muriel

Program in Mathematics, Graduate Center, City University of New York, 506 Fifth Avenue, NY 10016, United States

ARTICLE INFO

Article history:
Received 26 November 2015
Accepted 4 December 2015
Available online 19 December 2015

Keywords:
Time evolution equations
Arbitrary initial data
Field velocities
Pressure
Kinetic energy
Solutions of Navier–Stokes equation

ABSTRACT

We prescribe an alternative procedure for arriving at the time evolution equations for hydrodynamic local variables such as density, velocity fields, and kinetic energy using a time evolution equation of the single-particle distribution. It is suggested that when applied to various combinations of pair-potential between monoatoms, geometry and initial data, the prescription will have wide applications in hydrodynamics, including solutions of the Navier–Stokes equation.

1. Introduction

There have been a series of four recent papers that showed related solutions of the Navier–Stokes equation [1,2,3,4]. The first [1] was an existence proof of solutions in 3D. The second one [2] was a simplification of Muriel [1] to 2D with field velocities wrapped around a globe, producing flows symmetric about the equator. The third one [3] uses spherical symmetry to produce an analytic model of implosion toward the possibility of controlled nuclear fusion. The first three papers assumed an initially uniform system with delta-function momentum distributions. The fourth paper [4] is valid for an initial spatially uniform system and arbitrary initial momentum distribution. This paper generalizes the development for initial arbitrary space and momentum distributions, producing the most general approach to date.

We follow definitions and conventions in Muriel and Dresden [5] but review them for clarity and consistency with previous results.

Let $f(r,p,t)$ be the single-particle distribution function of a many-body system as in kinetic theory. It represents the probability that a particle in location r possesses the momentum p at time t. We use the phase space variables r,p in keeping with kinetic theory. Later, we will replace the momentum divided by the particle mass m with velocity to conform to the Navier–Stokes notation. The following time evolution equation for the single-particle distribution was derived in Muriel and Dresden [5]:

$$\begin{aligned} f(r,p,t) = {} & f\left(r-\tfrac{pt}{m}.p,0\right) \\ & + n_o \int_0^t ds_1 e^{-s_1 L_o} \int dr' \tfrac{\partial V(r-r')}{\partial r} \tfrac{\partial}{\partial p} \left(\tfrac{p}{m} \tfrac{\partial}{\partial r} f_2(r,r',p,0)\right) \\ & + n_o \int_0^t ds_1 \int_0^{s_1} ds_2 e^{-s_2 L_o} \int dr' \tfrac{\partial V(r-r')}{\partial r} \tfrac{\partial}{\partial p} \left(\tfrac{p}{m} \tfrac{\partial}{\partial r} f_2(r,r',p,0)\right) \\ & - n_o \int_0^t ds_1 \int_0^{s_1} ds_2 e^{-s_2 L_o} \int dp' \int dr' \tfrac{\partial V(r-r')}{\partial r} \tfrac{\partial}{\partial p} \left(\tfrac{p'}{m} \tfrac{\partial}{\partial r} f_2(r,r',p,p',0)\right) \\ & + \tfrac{n_o^2}{2} \int_0^t ds_1 \int_0^{s_1} ds_2 e^{-s_2 L_o} \int dr' \int dr'' \tfrac{\partial}{\partial p} \tfrac{\partial V(r-r')}{\partial r} \tfrac{\partial}{\partial p} \left(\tfrac{\partial V(r-r'')}{\partial r} f_3(r,r',r'',p,0)\right) \\ & + \tfrac{n_o^2}{2} \int_0^t ds_1 \int_0^{s_1} ds_2 e^{-s_2 L_o} \int dr' \tfrac{\partial}{\partial p} \tfrac{\partial V(r-r')}{\partial r} \tfrac{\partial}{\partial p} \left(\tfrac{\partial V(r-r')}{\partial r} f_2(r,r',p,0)\right) \\ & + \sum_{n=3}^{\infty} O\left(\tfrac{\partial^n}{\partial p^n}\right) \end{aligned} \tag{1}$$

where $f_2(r,r',p,0)$ is the mixed probability that two particles are in r,r' and the first particle has momentum p at time $t=0$. $f_2(r,r',p,p',0)$ is the mixed probability that two particles at r,r' each possess momentum p,p' at time $t=0$. $f_3(r,r',r'',p,0)$ is the mixed probability that particles are located at r,r',r'' where the first particle has momentum p at time $t=0$. These mixed probabilities come from the original formulation of the many-body problem from the Liouville equation. The mixed probability distributions, representing particle correlations at $t=0$ will be simplified to uncorrelated functions represented by simple products. We use the operator $L_o = \frac{p}{m}\frac{\partial}{\partial r}$. n_o is the average particle density. $V(r-r'')$ is the pair-potential of two particles located at r,r''. The existence of this pair-potential distinguishes this approach from the usual continuum model.

E-mail addresses: amuriel@gc.cuny.edu, amadormuriel@gmail.com

http://dx.doi.org/10.1016/j.rinp.2015.12.004
2211-3797/

We will calculate averages of $(1, p, p^2)$, so that using integration by parts, the contribution of the last term is zero, effectively truncating the series to a finite number of terms due to the vanishing of the momentum distribution at the boundary of the momentum space.

Using factored initial distributions, that is, $f_2(r, r') = f_1(r)f_1(r')$, etc., we rewrite Eq. (1) as

$$\begin{aligned}
&f(r,p,t) = f(r - \tfrac{pt}{m}, 0)\varphi(p,0)\\
&\quad + n_o \int_0^t ds_1 e^{-s_1 L_o} \int dr' \left[V(r-r') \tfrac{\partial}{\partial r'_j} f(r',0)\right] \tfrac{\partial}{\partial p_j}\left(\tfrac{p_i}{m}\tfrac{\partial}{\partial r_i} f(r,0)\varphi(p,0)\right)\\
&\tfrac{\partial V(r-r')}{\partial r} + n_o \int_0^t ds_1 \int_0^{s_1} ds_2 e^{-s_2 L_o} \int dr' \left[V(r-r') \tfrac{\partial}{\partial r'_j} f(r',0)\right] \tfrac{\partial}{\partial p_j}\left(\tfrac{p_i}{m}\tfrac{\partial}{\partial r_i} f(r,0)\varphi(p,0)\right)\\
&- n_o \int_0^t ds_1 \int_0^{s_1} ds_2 e^{-s_2 L_o} \int dr' V(r-r') \tfrac{\partial}{\partial r'_j} f(r',0)\\
&\quad \times \left(\tfrac{\partial}{\partial r_i} f(r,0) \tfrac{\partial}{\partial p_j}\varphi(p,0)\right) \int dp' \tfrac{p'_i}{m}\varphi(p',0)\\
&+ \tfrac{n_o^2}{2} \int_0^t ds_1 \int_0^{s_1} ds_2 e^{-s_2 L_o} f(r) \int dr' V(r-r') \tfrac{\partial f(r')}{\partial r'_j} \int dr'' V(r-r'') \tfrac{\partial f(r'')}{\partial r''_i} \left(\tfrac{\partial}{\partial p_j}\tfrac{\partial}{\partial p_i}\varphi(p)\right)\\
&+ \tfrac{n_o^2}{2} \int_0^t ds_1 \int_0^{s_1} ds_2 e^{-s_2 L_o} f(r) \int dr' (V(r-r'))^2 \tfrac{\partial}{\partial r'_j}\tfrac{\partial}{\partial r'_i} f(r') \tfrac{\partial}{\partial p_j}\tfrac{\partial}{\partial p_i}\varphi(p)
\end{aligned} \tag{2}$$

In Eq. (2), the symbols $r = (x, y, z), p = (p_x, p_y, p_z)$ are standard. In Cartesian component form, use summation over repeated indices,

2. Reduction procedure

We rewrite the individual terms of Eq. (2) in explicit Cartesian dot products, useful for evaluating future applications.

In simplifying the expressions of Eq. (2), we use the following properties:

(a) $\frac{\partial V(r-r')}{\partial r} = -\frac{\partial V(r-r')}{\partial r'}$, Newton's Third Law of action and reaction;
(b) integration by parts over r'; and
(c) $\int dr' \frac{\partial}{\partial r'_j}[V(r-r')f(r',0)] = 0$;

a boundary condition we use for the first time. Because of (c) we have departed from the space integrals over a cube of the earlier papers. Property (c) is applicable to most geometries.

We analyze each of the terms in Eq. (2):

Zeroth term

$$F\left(x - \frac{p_x t}{m}, y - \frac{p_y t}{m}, z - \frac{p_z t}{m}, 0\right)\varphi(p_x, p_x, p_x, 0)$$

First term

$$\int_0^t ds_1 e^{-s_1 L_o} S1$$

$$\begin{aligned}
S1 = &\int dr' \left[V(r-r') \tfrac{\partial}{\partial x'} f(r',0)\right] \tfrac{\partial}{\partial p_x}\left(\tfrac{p_x}{m}\tfrac{\partial}{\partial x} f(r,0)\varphi(p,0)\right)\\
&+ \int dr' \left[V(r-r') \tfrac{\partial}{\partial x'} f(r',0)\right] \tfrac{\partial}{\partial p_x}\left(\tfrac{p_y}{m}\tfrac{\partial}{\partial y} f(r,0)\varphi(p,0)\right)\\
&+ \int dr' \left[V(r-r') \tfrac{\partial}{\partial x'} f(r',0)\right] \tfrac{\partial}{\partial p_x}\left(\tfrac{p_z}{m}\tfrac{\partial}{\partial z} f(r,0)\varphi(p,0)\right)\\
&+ \int dr' \left[V(r-r') \tfrac{\partial}{\partial y'} f(r',0)\right] \tfrac{\partial}{\partial p_y}\left(\tfrac{p_x}{m}\tfrac{\partial}{\partial x} f(r,0)\varphi(p,0)\right)\\
&+ \int dr' \left[V(r-r') \tfrac{\partial}{\partial y'} f(r',0)\right] \tfrac{\partial}{\partial p_y}\left(\tfrac{p_y}{m}\tfrac{\partial}{\partial y} f(r,0)\varphi(p,0)\right)\\
&+ \int dr' \left[V(r-r') \tfrac{\partial}{\partial y'} f(r',0)\right] \tfrac{\partial}{\partial p_y}\left(\tfrac{p_z}{m}\tfrac{\partial}{\partial z} f(r,0)\varphi(p,0)\right)\\
&\int dr' \left[V(r-r') \tfrac{\partial}{\partial z'} f(r',0)\right] \tfrac{\partial}{\partial p_z}\left(\tfrac{p_x}{m}\tfrac{\partial}{\partial x} f(r,0)\varphi(p,0)\right)\\
&+ \int dr' \left[V(r-r') \tfrac{\partial}{\partial z'} f(r',0)\right] \tfrac{\partial}{\partial p_z}\left(\tfrac{p_y}{m}\tfrac{\partial}{\partial y} f(r,0)\varphi(p,0)\right)\\
&+ \int dr' \left[V(r-r') \tfrac{\partial}{\partial z'} f(r',0)\right] \tfrac{\partial}{\partial p_z}\left(\tfrac{p_z}{m}\tfrac{\partial}{\partial z} f(r,0)\varphi(p,0)\right)
\end{aligned} \tag{3}$$

Second term

$$+n_o \int_0^t ds_1 \int_0^{s_1} ds_2 e^{-s_2 L_o} S1$$

The first and second terms differ only in the time integrals.

Third term

$$-n_o \int_0^t ds_1 \int_0^{s_1} ds_2 e^{-s_2 L_o} S3$$

$$\begin{aligned}
S3 = &\int dr' \left[V(r-r') \tfrac{\partial}{\partial x'} f(r',0)\right] \tfrac{\partial}{\partial p_x}\left(\tfrac{p_x}{m}\tfrac{\partial}{\partial x} f(r,0)\varphi(p,0)\right) \int dp' \tfrac{p'_x}{m}\varphi(p',0)\\
&+ \int dr' \left[V(r-r') \tfrac{\partial}{\partial x'} f(r',0)\right] \tfrac{\partial}{\partial p_x}\left(\tfrac{p_y}{m}\tfrac{\partial}{\partial y} f(r,0)\varphi(p,0)\right) \int dp' \tfrac{p'_y}{m}\varphi(p',0)\\
&+ \int dr' \left[V(r-r') \tfrac{\partial}{\partial x'} f(r',0)\right] \tfrac{\partial}{\partial p_x}\left(\tfrac{p_z}{m}\tfrac{\partial}{\partial z} f(r,0)\varphi(p,0)\right) \int dp' \tfrac{p'_z}{m}\varphi(p',0)\\
&+ \int dr' \left[V(r-r') \tfrac{\partial}{\partial y'} f(r',0)\right] \tfrac{\partial}{\partial p_y}\left(\tfrac{p_x}{m}\tfrac{\partial}{\partial x} f(r,0)\varphi(p,0)\right) \int dp' \tfrac{p'_x}{m}\varphi(p',0)\\
&+ \int dr' \left[V(r-r') \tfrac{\partial}{\partial y'} f(r',0)\right] \tfrac{\partial}{\partial p_y}\left(\tfrac{p_y}{m}\tfrac{\partial}{\partial y} f(r,0)\varphi(p,0)\right) \int dp' \tfrac{p'_y}{m}\varphi(p',0)\\
&+ \int dr' \left[V(r-r') \tfrac{\partial}{\partial y'} f(r',0)\right] \tfrac{\partial}{\partial p_y}\left(\tfrac{p_z}{m}\tfrac{\partial}{\partial z} f(r,0)\varphi(p,0)\right) \int dp' \tfrac{p'_z}{m}\varphi(p',0)\\
&\int dr' \left[V(r-r') \tfrac{\partial}{\partial z'} f(r',0)\right] \tfrac{\partial}{\partial p_z}\left(\tfrac{p_x}{m}\tfrac{\partial}{\partial x} f(r,0)\varphi(p,0)\right) \int dp' \tfrac{p'_x}{m}\varphi(p',0)\\
&+ \int dr' \left[V(r-r') \tfrac{\partial}{\partial z'} f(r',0)\right] \tfrac{\partial}{\partial p_z}\left(\tfrac{p_y}{m}\tfrac{\partial}{\partial y} f(r,0)\varphi(p,0)\right) \int dp' \tfrac{p'_y}{m}\varphi(p',0)\\
&+ \int dr' \left[V(r-r') \tfrac{\partial}{\partial z'} f(r',0)\right] \tfrac{\partial}{\partial p_z}\left(\tfrac{p_z}{m}\tfrac{\partial}{\partial z} f(r,0)\varphi(p,0)\right) \int dp' \tfrac{p'_z}{m}\varphi(p',0)
\end{aligned} \tag{4}$$

Fourth term

$$\frac{n_o^2}{2} \int_0^t ds_1 \int_0^{s_1} ds_2 e^{-s_2 L_o} f(r) S4$$

where

$$\begin{aligned}
S4 = &\int dr' V(r-r') \tfrac{\partial f(r')}{\partial x'} \int dr'' V(r-r'') \tfrac{\partial f(r'')}{\partial x''} \left(\tfrac{\partial}{\partial p_x}\tfrac{\partial}{\partial p_x}\varphi(p)\right)\\
&+ \int dr' V(r-r') \tfrac{\partial f(r')}{\partial y'} \int dr'' V(r-r'') \tfrac{\partial f(r'')}{\partial x''} \left(\tfrac{\partial}{\partial p_y}\tfrac{\partial}{\partial p_x}\varphi(p)\right)\\
&+ \int dr' V(r-r') \tfrac{\partial f(r')}{\partial z'} \int dr'' V(r-r'') \tfrac{\partial f(r'')}{\partial x''} \left(\tfrac{\partial}{\partial p_z}\tfrac{\partial}{\partial p_x}\varphi(p)\right)\\
&+ \int dr' V(r-r') \tfrac{\partial f(r')}{\partial x'} \int dr'' V(r-r'') \tfrac{\partial f(r'')}{\partial y''} \left(\tfrac{\partial}{\partial p_x}\tfrac{\partial}{\partial p_y}\varphi(p)\right)\\
&+ \int dr' V(r-r') \tfrac{\partial f(r')}{\partial y'} \int dr'' V(r-r'') \tfrac{\partial f(r'')}{\partial y''} \left(\tfrac{\partial}{\partial p_y}\tfrac{\partial}{\partial p_y}\varphi(p)\right)\\
&+ \int dr' V(r-r') \tfrac{\partial f(r')}{\partial z'} \int dr'' V(r-r'') \tfrac{\partial f(r'')}{\partial y''} \left(\tfrac{\partial}{\partial p_z}\tfrac{\partial}{\partial p_y}\varphi(p)\right)\\
&+ \int dr' V(r-r') \tfrac{\partial f(r')}{\partial x'} \int dr'' V(r-r'') \tfrac{\partial f(r'')}{\partial z''} \left(\tfrac{\partial}{\partial p_x}\tfrac{\partial}{\partial p_z}\varphi(p)\right)\\
&+ \int dr' V(r-r') \tfrac{\partial f(r')}{\partial y'} \int dr'' V(r-r'') \tfrac{\partial f(r'')}{\partial z''} \left(\tfrac{\partial}{\partial p_y}\tfrac{\partial}{\partial p_z}\varphi(p)\right)\\
&+ \int dr' V(r-r') \tfrac{\partial f(r')}{\partial z} \int dr'' V(r-r'') \tfrac{\partial f(r'')}{\partial z''} \left(\tfrac{\partial}{\partial p_z}\tfrac{\partial}{\partial p_z}\varphi(p)\right)
\end{aligned} \tag{5}$$

where we now suppress the $t = 0$ qualification for all initial data henceforth.

Fifth term

$$+\frac{n_o^2}{2} \int_0^t ds_1 \int_0^{s_1} ds_2 e^{-s_2 L_o} f(r) S5$$

$$\begin{aligned}
S5 = &\int dr' (V(r-r'))^2 \tfrac{\partial}{\partial x'}\tfrac{\partial}{\partial x'} f(r') \tfrac{\partial}{\partial p_x}\tfrac{\partial}{\partial p_x}\varphi(p)\\
&+ \int dr' (V(r-r'))^2 \tfrac{\partial}{\partial y'}\tfrac{\partial}{\partial x'} f(r') \tfrac{\partial}{\partial p_y}\tfrac{\partial}{\partial p_x}\varphi(p)\\
&+ \int dr' (V(r-r'))^2 \tfrac{\partial}{\partial z'}\tfrac{\partial}{\partial x'} f(r') \tfrac{\partial}{\partial p_z}\tfrac{\partial}{\partial p_x}\varphi(p)\\
&+ \int dr' (V(r-r'))^2 \tfrac{\partial}{\partial x'}\tfrac{\partial}{\partial y'} f(r') \tfrac{\partial}{\partial p_x}\tfrac{\partial}{\partial p_y}\varphi(p)\\
&+ \int dr' (V(r-r'))^2 \tfrac{\partial}{\partial y'}\tfrac{\partial}{\partial y'} f(r') \tfrac{\partial}{\partial p_y}\tfrac{\partial}{\partial p_y}\varphi(p)\\
&+ \int dr' (V(r-r'))^2 \tfrac{\partial}{\partial z'}\tfrac{\partial}{\partial y'} f(r') \tfrac{\partial}{\partial p_z}\tfrac{\partial}{\partial p_y}\varphi(p)\\
&+ \int dr' (V(r-r'))^2 \tfrac{\partial}{\partial x'}\tfrac{\partial}{\partial z'} f(r') \tfrac{\partial}{\partial p_x}\tfrac{\partial}{\partial p_z}\varphi(p)\\
&+ \int dr' (V(r-r'))^2 \tfrac{\partial}{\partial y'}\tfrac{\partial}{\partial z'} f(r') \tfrac{\partial}{\partial p_y}\tfrac{\partial}{\partial p_z}\varphi(p)\\
&+ \int dr' (V(r-r'))^2 \tfrac{\partial}{\partial z'}\tfrac{\partial}{\partial z'} f(r') \tfrac{\partial}{\partial p_z}\tfrac{\partial}{\partial p_z}\varphi(p)
\end{aligned} \tag{6}$$

In [1–4], for an initially uniform system, only the zeroth term and the fifth term are non-zero. Here, the most general initial data activates the first, second, third and fourth term of Eq. (2)