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On some combinations of terms of a recurrence sequence

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ABSTRACT

Let $(G_m)_{m\geq 0}$ be an integer linear recurrence sequence (under some weak technical conditions) and let $x \geq 1$ be an integer. In this paper, we are interested in the problem of finding combinations of the form $xG_n + G_{n-1}$ which belongs to $(G_m)_{m\geq 0}$ for infinitely many positive integers n. In this case, we shall make explicit an upper bound for x which only depends on the roots of the characteristic polynomial of this recurrence. As application, we shall study the k-nacci case.

1. Introduction

A sequence $(G_n)_{n\geq 0}$ is *a linear recurrence sequence* with coefficients $c_0, c_1,\ldots,c_{k-1}$, with $c_0 \neq 0$, if

$$G_{n+k} = c_{k-1}G_{n+k-1} + \cdots + c_1G_{n+1} + c_0G_n, \tag{1}$$

for all positive integer n. A recurrence sequence is therefore completely determined by the *initial values* $G_0, \ldots, G_{k-1}$, and by the coefficients $c_0, c_1, \ldots, c_{k-1}$. The integer k is called the *order* of the linear recurrence. The *characteristic polynomial* of the sequence $(G_n)_{n\geq 0}$ is given by

$$\psi(x) = x^k - c_{k-1}x^{k-1} - \cdots - c_1x - c_0.$$

It is well-known that for all n

$$G_n = g_1(n)\alpha_1^n + \cdots + g_\ell(n)\alpha_\ell^n, \tag{2}$$

where α_j is a root of $\psi(x)$ and $g_j(x)$ is a polynomial over a certain number field, for $j = 1, \ldots, \ell$. In this paper, we consider only integer recurrence sequences, i.e., recurrence sequences whose coefficients and initial values are integers. Hence, $g_j(n)$ is an algebraic number, for all $j = 1, \ldots, \ell$, and $n \in \mathbb{Z}$.

Possibly, the most known recurrence sequence is the *Fibonacci sequence* $(F_n)_{n\geq 0}$ defined by the recurrence $F_{n+1} = F_n + F_{n-1}$ ($n \geq 1$) with initial values $F_0 = 0$ and $F_1 = 1$. Its companion sequence is the sequence of Lucas numbers $(L_n)_{n\geq 0}$ which are defined by the same recurrence but with initial values $L_0 = 2$ and $L_1 = 1$.

The Fibonacci numbers are known for their amazing properties (see [7] for the history, properties, and rich applications of the Fibonacci sequence and some of its variants). Among the several pretty algebraic identities involving these numbers, we are interested in the following one

$$xF_n + F_{n-1} = F_{n+x}, \quad n \geq 1 \quad \text{and} \quad x \in \{1, 2\}. \tag{3}$$

In particular, this naive identity (which is completely easy to prove) tell us, in particular (case $x = 2$), that the double of a Fibonacci number added by its preceding term is always a Fibonacci number. So, natural questions arise: Does the same property hold for $3F_n + F_{n-1}$? And for $4F_n + F_{n-1}$? And so on? This paper will be motivated by these questions.

Thus, the aim of this work is to combine some Diophantine tools (asymptotic estimates and Galois theory) in order to study the possibilities of existence of such identities in the case of a general linear recurrence. In particular, we show that it is possible to obtain an upper bound for x in the case when $xG_n + G_{n-1}$ belongs to the sequence $(G_m)_{m\geq 0}$ for infinitely many integers m. This upper bound is effective and can be make explicit by means of the recurrence sequence. More precisely, our main result is the following.

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Theorem 1. *Let $(G_n)_{n\geq 0}$ be an integer linear recurrence (of order at least 2) such that its characteristic polynomial $\psi(x)$ has a simple positive root being the unique zero outside the unit circle. If $x \geq 1$ is an integer such that $xG_n + G_{n-1}$ belongs to $(G_n)_{n\geq 0}$ for infinitely many integers n, then*

$$x \leq \frac{2}{\max_{\psi(z)=0, |z|\leq 1} |z|}. \tag{4}$$

Let us give a brief overview of our strategy for proving Theorem 1. First, by supposing that $xG_n + G_{n-1} = G_{a(n)}$, we prove that $a(n) = n + a$, for some integer a (this is done by using lower and upper bounds for G_n). After, we use an asymptotic formula for G_{n+t}/G_n (when $t \in \{-1, a\}$) to obtain the equation $\alpha x + 1 = \alpha^{a+1}$, where α is the dominant root of the sequence $(G_m)_{m\geq 0}$. Since $\alpha > 1$, the right-hand side of the previous identity can be very large. However, by conjugating by some convenient automorphism of the Galois group of the characteristic polynomial of the recurrence, we get $|\beta x + 1| = |\beta|^{a+1} < 1$ (since we are supposing that the algebraic conjugates of α are inside the unit circle). In conclusion, we obtain an upper bound for x depending on β.

Before stating an application of our previous result, we need some definitions.

Let $k \geq 2$ and denote $F^{(k)} := (F_n^{(k)})_{n\geq -(k-2)}$, the *k-generalized Fibonacci sequence* whose terms satisfy the recurrence relation

$$F_{n+k}^{(k)} = F_{n+k-1}^{(k)} + F_{n+k-2}^{(k)} + \cdots + F_n^{(k)}, \tag{5}$$

with initial conditions $0, 0, \ldots, 0, 1$ (k terms) and such that the first nonzero term is $F_1^{(k)} = 1$.

The above sequence is one among the several generalizations of Fibonacci numbers. Such a sequence is also called *k-step Fibonacci sequence*, the *Fibonacci k-sequence*, or *k-nacci sequence*. Clearly for $k = 2$, we obtain the classical Fibonacci numbers, for $k = 3$, the Tribonacci numbers, for $k = 4$, the Tetranacci numbers, etc.

Recently, these sequences have been the main subject of many papers. We refer to [3] for results on the largest prime factor of $F_n^{(k)}$ and we refer to [1] for the solution of the problem of finding powers of two belonging to these sequences. In 2013, two conjectures concerning these numbers were proved. The first one, proved by Bravo and Luca [4] is related to *repdigits* (i.e., numbers with only one distinct digit in its decimal expansion) among k-nacci numbers (proposed by Marques [9]) and the second one, a conjecture (proposed by Noe and Post [10]) about coincidences between terms of these sequences, proved independently by Bravo–Luca [2] and Marques [8].

As application of our Theorem 1, we shall solve completely the case when $G = F^{(k)}$. More precisely, we proved that

Theorem 2. *The only pairs $(x, k) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 2}$ such that*

$$xF_n^{(k)} + F_{n-1}^{(k)} \in \left(F_m^{(k)}\right)_{m\geq 0} \tag{6}$$

for infinitely many positive integers n are $(x, k) = (1, 2)$ and $(2, 2)$.

We remark the existence of the interesting identities of the form

$$F_{n-k+1}^{(k)} + x\left(F_{n-k+2}^{(k)} + \cdots + F_n^{(k)}\right) = F_{n+k}^{(k)} \tag{7}$$

for all $n \geq 1, k \geq 2$ and $x \in \{1, 2\}$. This also can be deduce from the previous theorems.

We point out that the main novelty of our results is that it is very general and, in fact, it predicts when it is possible to obtain identities of the form $xG_n + G_{n-1} = G_m$ for a very large class of recurrence sequences. Also, our method and our results can be interesting for other authors which work on these kinds of recurrences.

2. General recurrence sequences

2.1. Auxiliary results

In this section, we recall some results that will be very useful for the proof of the above theorems. Let $\psi(x)$ be the characteristic polynomial of a linear recurrence $(G_n)_{n\geq 0}$. One can factor $\psi(x)$ over the set of complex numbers as

$$\psi(x) = (x - \alpha_1)^{m_1}(x - \alpha_2)^{m_2} \cdots (x - \alpha_t)^{m_t}, \tag{8}$$

where $\alpha_1, \ldots, \alpha_t$ are distinct non-zero complex numbers (called the *roots* of the recurrence) and $m_1, \ldots, m_t$ are positive integers. A fundamental result in the theory of recurrence sequences asserts that there exist uniquely determined non-zero polynomials $g_1, \ldots, g_\ell \in \mathbb{Q}(\{\alpha_j\}_{j=1}^t)[x]$, with $\deg g_j \leq m_j - 1$, for $j = 1, \ldots, t$, such that

$$G_n = g_1(n)\alpha_1^n + \cdots + g_t(n)\alpha_t^n, \quad \text{for all} \quad n. \tag{9}$$

For more details, see [11, Theorem C.1]. A root α_j of the recurrence is called a *dominant root* if $|\alpha_j| > |\alpha_i|$, for all $j \neq i \in \{1, \ldots, t\}$. The corresponding polynomial $g_j(n)$ is named the *dominant polynomial* of the recurrence.

In the case of the Fibonacci sequence, the above formula is known as *Binet's formula*:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{10}$$

where $\alpha = (1 + \sqrt{5})/2$ (the golden number) and $\beta = (1 - \sqrt{5})/2 = -1/\alpha$.

Now, we shall prove some lemmas which will be essential ingredients in what follows. Throughout the paper, α_1 will denote the dominant root of $(G_n)_{n\geq 0}$.

Lemma 1. *Let a be any integer and let $(G_n)_{n\geq 0}$ be any linear integer sequence satisfying the hypotheses of the Theorem 1. Then*

$$\lim_{n\to\infty} \frac{G_{n+a}}{G_n} = \alpha_1^a. \tag{11}$$

Proof. Since α_1 is simple root, we have immediately $m_1 = 1$ and then the degree of dominant polynomial is at most $m_1 - 1 = 0$, so it is a constant, say g_1.

Now, we use the formula in (9) to obtain

$$\lim_{n\to\infty} \frac{G_{n+a}}{G_n} = \lim_{n\to\infty} \frac{g_1\alpha_1^{n+a} + g_2(n+a)\alpha_2^{n+a} + \cdots + g_t(n+a)\alpha_t^{n+a}}{g_1\alpha_1^n + g_2(n)\alpha_2^n + \cdots + g_t(n)\alpha_t^n}$$

$$= \lim_{n\to\infty} \frac{\alpha_1^a + \frac{g_2(n+a)}{g_1\alpha_1^n}\alpha_2^{n+a} + \cdots + \frac{g_t(n+a)}{g_1\alpha_1^n}\alpha_t^{n+a}}{1 + \frac{g_2(n)}{g_1\alpha_1^n}\alpha_2^n + \cdots + \frac{g_t(n)}{g_1\alpha_1^n}\alpha_t^n} = \alpha_1^a,$$