Chaos, Solitons and Fractals 87 (2016) 47–50

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Infinitely many homoclinic solutions for a second-order Hamiltonian system with locally defined potentials[☆]

Xiaoping Wang[*]

Department of Mathematics, Xiangnan College, Chenzhou, Hunan 423000, P.R. China

ARTICLE INFO

Article history:
Received 1 September 2015
Revised 20 February 2016
Accepted 26 February 2016
Available online 21 March 2016

2000 MSC:
34C37
58E05
70H05

Keywords:
Homoclinic solution
Hamiltonian system
Symmetric Mountain Pass Theorem

ABSTRACT

In this paper, we study homoclinic solutions of the following second-order Hamiltonian system

$\ddot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0,$

where $t \in \mathbb{R}, u \in \mathbb{R}^N$, $L : \mathbb{R} \to \mathbb{R}^{N\times N}$ and $W : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$. Applying a new symmetric Mountain Pass Theorem established by Kajikiya, we prove the existence of infinitely many homoclinic solutions for the above system in the case where $L(t)$ is coercive but unnecessarily positive definite for all $t \in \mathbb{R}$, and $W(t, x)$ is only locally defined near the origin with respect to x. Our results significantly generalize and improve related ones in the literature.

© 2016 Published by Elsevier Ltd.

1. Introduction

Consider the second-order Hamiltonian system

$$\ddot{u} - L(t)u + \nabla W(t, u) = 0, \tag{1.1}$$

where $t \in \mathbb{R}, u \in \mathbb{R}^N$, $L : \mathbb{R} \to \mathbb{R}^{N\times N}$, $W : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $\nabla W(t, x) = \nabla_x W(t, x)$. Here, as usual, we say that a solution $u(t)$ of (1.1) is homoclinic (to 0) if $u(t) \to 0$ as $t \to \pm\infty$. In addition, if $u(t) \not\equiv 0$ then $u(t)$ is called a nontrivial homoclinic solution.

In the last several decades, the existence and multiplicity of homoclinic solutions for system (1.1) has been extensively investigated via critical point theory. See, for example, [1–6,8–21,23–28,31,32] and the references therein. However, we emphasize that in all these papers $W(t, x)$ was always required to satisfy some kind of growth conditions at infinity with respect to x, such as superquadratic, asymptotically quadratic or subquadratic growth.

In recent paper, Zhang and Chu [29] studied the existence of infinitely many homoclinic solutions for (1.1) in the case where $L(t)$ is coercive but unnecessarily positive definite for all $t \in \mathbb{R}$, and $W(t, x)$ is only locally defined near the origin with respect to x. More precisely, they presented the following assumptions:

(H0) $L \in C(\mathbb{R}, \mathbb{R}^{N\times N})$, $L(t)$ is a symmetric matrix for all $t \in \mathbb{R}$, and there exists a constant $\nu < 2$ such that $\lim_{|t|\to\infty} |t|^{2-\nu} l(t) = \infty$, where

$$l(t) := \inf_{x\in\mathbb{R}^N, |x|=1} (L(t)x, x);$$

(H1) $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, $W(t, 0) \equiv 0$, and there exist constants $c_0 > 0$, $\delta > 0$ and $\max\{3/2, (4-\nu)/(3-\nu)\} < \kappa < 2$ such that

$$|\nabla W(t, x)| \le c_0 |x|^{\kappa-1}, \quad \forall\ (t, x) \in \mathbb{R} \times B_\delta(0); \tag{1.2}$$

(H2) $\lim_{|x|\to 0} \frac{W(t,x)}{|x|^2} = \infty$ uniformly $t \in \mathbb{R}$;

(H3) $2W(t, x) - (\nabla W(t, x), x) > 0, \ \forall\ (t, x) \in \mathbb{R} \times (B_\delta(0) \setminus \{0\})$;

(H4) $W(t, -x) = W(t, x)$ for $(t, x) \in \mathbb{R} \times B_\delta(0)$.

Since $L(t)$ is not uniformly positive definite, the spectral of the operator $-\frac{d^2}{dt^2} + L(t)$ may contain negative numbers and zero. The energy functional associated with system (1.1) is indefinite, i.e., it is bounded neither from below nor from above. The main difficulty in [29] is how to prove the boundedness of the Palais–Smale sequence.

We note that (H0)–(H3) imply that there exists a constant $a_0 > 0$ such that $L(t) + 2a_0 I_N$ is uniformly positive definite for all $t \in \mathbb{R}$, and $W(t, x) + a_0|x|^2$ still satisfy (H1)–(H3). It is evident that (1.1) is equivalent to the following system:

$$\ddot{u} - [L(t) + 2a_0 I_N]u + \nabla[W(t, u) + a_0|u|^2] = 0. \tag{1.3}$$

☆ This work is partially supported by the NNFC (No: 11471278) of China.

* Tel.: 135413328831.
E-mail address: wxp31415@sina.com

http://dx.doi.org/10.1016/j.chaos.2016.02.034
0960-0779/© 2016 Published by Elsevier Ltd.

Following partially the idea of [7] in dealing with the Dirichlet boundary problems, we will first modify $W(t, x)$ for x outside a neighborhood of the origin 0 to get $\widehat{W}(t,x)$ and introduce the following modified Hamiltonian system

$$\ddot{u} - [L(t) + 2a_0 I_N]u + \nabla[\widehat{W}(t,u) + a_0|u|^2] = 0, \tag{1.4}$$

where $\widehat{W}$ will be specified in Section 2. For system (1.4), the energy functional associated with it is even and bounded from below. Hence, we can use a new symmetric mountain pass lemma obtained in [7] to show that system (1.4) possesses a sequence of homoclinic solutions, which converges to zero in L^∞ norm. Consequently, we obtain infinitely many homoclinic solutions for system (1.1), see [29,30].

Before presenting our theorem, we introduce the following assumptions:

(L1) $L \in C(\mathbb{R}, \mathbb{R}^{N\times N})$ such that $L(t)$ is a symmetric matrix for all $t \in \mathbb{R}$ and $\inf_{\mathbb{R}} l(t) > -\infty$;

(L2) There exists a constant $\nu < 2$ such that

$$\mathrm{meas}\left\{t \in \mathbb{R} : \ |t|^{-\nu}L(t) \ngeq MI_N\right\} < \infty, \qquad \forall\ M > 0,$$

where meas(·) denotes the Lebesgue measure in $\mathbb{R}$;

(W1) $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, $W(t, 0) \equiv 0$, and there exist constants $c_0 > 0$, $\delta > 0$ and $\max\{1, 2/(3-\nu)\} < \kappa < 2$ such that

$$|\nabla W(t,x)| \le c_0|x|^{\kappa-1}, \quad \forall\ (t,x) \in \mathbb{R} \times B_\delta(0); \tag{1.5}$$

(W2) There exist a $t_0 \in \mathbb{R}$ and a constant $\eta > 0$ such that

$$\liminf_{|x|\to 0} \inf_{t\in[t_0-\eta, t_0+\eta]} \frac{W(t,x)}{|x|^2} > -\infty$$

and

$$\limsup_{|x|\to 0} \inf_{t\in[t_0-\eta, t_0+\eta]} \frac{W(t,x)}{|x|^2} = +\infty;$$

(W3) $W(t,-x) = W(t,x)$ for $(t,x) \in \mathbb{R} \times B_\delta(0)$.

Now, we are ready to state the main result of this paper.

Theorem 1.1. *Assume that L and W satisfy (L1), (L2), (W1), (W2) and (W3). Then system (1.1) possesses a sequence $\{u_n\}$ of homoclinic solutions such that $\|u_n\|_\infty \to 0$ as $n \to \infty$.*

Remark 1.2. A condition similar to (W2) on the nonlinearity W was first introduced in [7] for the Dirichlet boundary problems.

Remark 1.3. In our theorem, $L(t)$ is unnecessarily required to be either uniformly positive definite or coercive. For example $L(t) = (t^2|\sin t| - 1)I_N$ satisfies (L1) and (L2), but does not satisfy (H0). It is easy to check that the following functions W satisfy (W1), (W2) and (W3):

$$W(t,x) = \cos t\ |x|^{4/3} + \sin t\ |x|^p, \quad p > 4/3; \tag{1.6}$$

$$W(t,x) = \cos t \sin |x|^{3/2}. \tag{1.7}$$

One can see that they satisfy neither (H2) nor (H3).

2. Variational setting and some lemmas

Throughout this section, we make the following assumption instead of (L1):

(L1′) $L \in C(\mathbb{R}, \mathbb{R}^{N\times N})$, for all $t \in \mathbb{R}$, $L(t)$ is positive definite symmetric matrix and

$$(L(t)x, x) \ge |x|^2, \quad \forall\ (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

We work in the Hilbert space

$$E = \left\{u \in H^1(\mathbb{R}, \mathbb{R}^N) \ : \ \int_{\mathbb{R}} \left[|\dot{u}|^2 + (L(t)u,u)\right]\mathrm{d}t < +\infty\right\}$$

equipped with the inner product

$$\langle u, v\rangle = \int_{\mathbb{R}} [(\dot{u}, \dot{v}) + (L(t)u, v)]\mathrm{d}t, \quad u, v \in E,$$

which induces the norm

$$\|u\| = \left\{\int_{\mathbb{R}} \left[|\dot{u}|^2 + (L(t)u,u)\right]\mathrm{d}t\right\}^{1/2}, \quad u \in E.$$

Evidently, E is continuously embedded into $H^1(\mathbb{R}, \mathbb{R}^N)$ and hence continuously embedded into $L^p(\mathbb{R}, \mathbb{R}^N)$ for $2 \le p \le \infty$, i.e., there exists $\gamma_p > 0$ such that

$$\|u\|_p \le \gamma_p\|u\|, \quad \forall\ u \in E, \tag{2.1}$$

where $\|u\|_p$ denotes the usual norm in $L^p(\mathbb{R}, \mathbb{R}^N)$ for all $2 \le p \le \infty$.

Lemma 2.1. *[22] Under assumptions (L1′) and (L2), the embedding from E into $L^p(\mathbb{R})$ is compact for $1 \le p \in (2/(3-\nu), \infty]$.*

Choose $r \in (0, \delta/2)$. Define a cut-off function $\xi \in C^1(\mathbb{R}, \mathbb{R})$ satisfying

$$\xi(t) = \begin{cases} 1, & 0 \le t \le r, \\ 0, & t \ge 2r, \end{cases}$$

and $-2r \le \xi'(t) < 0$ for $r < t < 2r$. In view of (W1), one has

$$|W(t,x)| \le c_0|x|^\kappa, \quad \forall\ (t,x) \in \mathbb{R} \times B_{2r}(0). \tag{2.2}$$

Define

$$\widehat{W}(t,x) = \xi(|x|)W(t,x) + c_0[1-\xi(|x|)]|x|^\kappa, \quad \forall\ (t,x) \in \mathbb{R} \times \mathbb{R}^N. \tag{2.3}$$

Lemma 2.2. *Assume that (W1) holds. Then*

$$|\widehat{W}(t,x)| \le c_0|x|^\kappa, \quad \forall\ (t,x) \in \mathbb{R} \times \mathbb{R}^N \tag{2.4}$$

and

$$|\nabla\widehat{W}(t,x)| \le 11c_0|x|^{\kappa-1}, \quad \forall\ (t,x) \in \mathbb{R} \times \mathbb{R}^N. \tag{2.5}$$

Proof. It follows from (2.2) and (2.3) that (2.4) holds. By direct computation, one has

$$\nabla\widehat{W}(t,x) = \xi(|x|)\nabla W(t,x) + \frac{\xi'(|x|)}{|x|}W(t,x)x + c_0\kappa[1-\xi(|x|)]|x|^{\kappa-2}x - c_0\xi'(|x|)|x|^{\kappa-1}x,$$

which, together with (W1), (2.2) and $|\xi'(|x|)||x| \le 4$, implies (2.5) holds. □

Now we define a functional Φ on E by

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}} \left[|\dot{u}|^2 + (L(t)u,u)\right]\mathrm{d}t - \int_{\mathbb{R}} \widehat{W}(t,u)\mathrm{d}t. \tag{2.6}$$

By Lemmas 2.1 and 2.2, under assumptions (L1′), (L2) and (W1), the functional Φ is of class $C^1(E, \mathbb{R})$. Moreover,

$$\Phi(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} \widehat{W}(t,u)\mathrm{d}t, \quad \forall\ u \in E \tag{2.7}$$

and

$$\langle\Phi'(u), v\rangle = \langle u, v\rangle - \int_{\mathbb{R}} (\nabla\widehat{W}(t,u), v)\mathrm{d}t, \quad \forall\ u, v \in E. \tag{2.8}$$

Let X be a Banach space and A a subset of X. A is said to be symmetric if $u \in A$ implies $-u \in A$. Denote by Γ the family of all closed symmetric subset of X which does not contain 0. For any $A \subset \Gamma$, define the genus $\gamma(A)$ of A by the smallest integer k such

Download English Version:

https://daneshyari.com/en/article/1888740

Download Persian Version:

https://daneshyari.com/article/1888740

Daneshyari.com