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Stability of a spatial model of social interactions

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ABSTRACT

We study a spatial model of social interactions. Though the properties of the spatial equilibrium have been largely discussed in the existing literature, the stability of equilibrium remains an unaddressed issue. Our aim is to fill up this gap by introducing dynamics in the model and by determining the stability of equilibrium. First we derive a variational equation useful for the stability analysis. This allows to study the corresponding eigenvalue problem. While odd modes are shown to be always stable, there is a single even mode of which stability depends on the model parameters. Finally various numerical simulations illustrate our theoretical results.

1. Introduction

The economic literature on spatial agglomerations has been emphasizing the role of increasing returns in the production sector as favoring the spatial clustering of economic activities, see [1]. However, it is known that both market and non-market forces play an important role in determining the balance between agglomeration and dispersion forces in a spatial economy. In particular, social interactions through face-to-face contacts also contribute to the gathering of individuals in villages, agglomerations, or cities, see [2]. Beckmann [3] introduced social interactions into a land market model. In his model, the spatial equilibrium structure results from the interplay between the agglomeration force generated by social interactions and the dispersion force channeled by land prices. Beckmann's work has been revisited by Fujita and Thisse [4], Mossay and Picard [5], and Blanchet et al. [6] by studying further the properties of the spatial equilibrium. In particular, Mossay and Picard [5] have shown that Beckmann's equilibrium along a segment is unique and have extended the analysis along a circle. Blanchet et al. [6] have extended Beckmann's framework so as to encompass general agents' preferences: along a segment, the uniqueness of spatial equilibrium holds for a large class of utility functions. Though the static aspects of Beckmann's framework have been largely studied, dynamic aspects of the model have not received attention yet. The purpose of this paper is to study the stability of spatial equilibrium in Beckmann's model, an issue which is left unaddressed in the existing literature.

First, we extend the spatial model of social interactions by Mossay and Picard [5] to a dynamic setting accounting for the fact that individuals tend to relocate to locations providing them with higher utility levels. This leads to an integro-differential equation governing the evolution of the population distribution over space and time. In the New Economic Geography literature (see e.g. [1]), most models are often discrete and usually involve a small number of locations. In that case, stability methods require the study of a finite number of eigenvalues (e.g. [7,8]). In that literature, dynamic models set in continuous space, as is the case here, are rare. The few existing studies rely on the method of normal modes to analyze stability; e.g. [1,9,10] or [11]. In contrast here, we derive a variational equation for stability, and only then study the even and odd modes of the eigenvalue problem.

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Second, we present various numerical simulations illustrating our theoretical results. For this purpose we perform simulations for stable and unstable parameter configurations as well as for various initial conditions such as uniform random noise or multiple-center configurations.

Section 2 provides the dynamic extension of the model. The stability analysis is presented in Section 3. In Section 4, numerical simulations are performed and explained. Section 5 concludes.

2. The spatial model of social interactions

Our model builds on the spatial model of social interactions by Mossay and Picard [5], and Blanchet et al. [6] along a line segment. Let us denote the density of agents in location x at time t by $\lambda(x, t)$. Agents benefit from social contacts with other agents. In order to establish those contacts, agents have to travel along the segment. The social utility that an agent in location x derives from interacting with other agents is given by

$$S(x,t) = \alpha \int \lambda(y,t)\, dy - \tau \int |x-y|\, \lambda(y,t)\, dy \tag{1}$$

where the first integral describes the social interactions with other agents with $\alpha > 0$ and the second one accounts for the traveling cost incurred to meet them with $\tau > 0$. The utility $V(x, t)$ of agents consists of the social utility minus a disutility resulting from congestion

$$V(x,t) = S(x,t) - \beta\, \lambda(x,t) \tag{2}$$

where $\beta\lambda(x, t)$ is the congestion cost with $\beta > 0$. In [5], the congestion cost results from congestion in the land market: higher agent densities lead to higher land prices, which translates into a disutility. In that paper, the agents' preference for land is chosen so that the resulting congestion cost is linear in λ, which is the functional form we also retain in this paper.

We now extend the static framework established by Mossay and Picard to a dynamic setting. In our model, agents tend to relocate towards locations providing them with higher utilities:

$$\frac{\partial \lambda}{\partial t} = k\,[V(x,t) - \bar{V}(t)]\, \lambda(x,t) \tag{3}$$

where $\bar{V}(t)$ denotes the first spatial moment of the utility, $\int\lambda(y, t)V(y, t)\, dy$, and $k > 0$ a mobility parameter.

2.1. Invariant manifold

By integrating Eq. (3) over the whole domain and by denoting :

$$I = \int \lambda(y,t)\, dy\,, \tag{4}$$

we obtain the following equation for I:

$$\frac{dI}{dt} = k\bar{V}(t)\,[1-I]\,, \tag{5}$$

we see that $I = 1$ is an invariant manifold associated with the dynamics given by Eq. (3). The transverse stability of this manifold is given by the sign of $\bar{V}$. If $\bar{V} > 0$ the manifold is locally stable. We will return to this point latter.

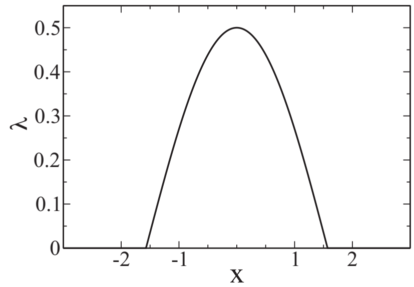

Fig. 1. Steady solution $\lambda^{(s)}(x)$ for $\alpha = 1$, $\beta = 2$ and $\tau = 1$ such that $\delta = 1$ and $b = \pi/2$.

2.2. Steady-state solution

Let us now look for a steady-state solution to Eq. (3). If such a solution exists it satisfies:

$$V(x,t) = \bar{V}(t) = \text{constant} \tag{6}$$

or equivalently

$$\tau \int |x-y|\, \lambda^{(s)}(y)\, dy + \beta\, \lambda^{(s)}(x) = \text{constant} \tag{7}$$

where the superscript s denotes the steady-state solution. By construction, the steady-state solution of our model corresponds to the spatial equilibrium studied in [5]. Without loss of generality, we restrict the search for a steady-state solution on finite support. Therefore, we can assume that $\lambda^{(s)}(x)$ is centered around $x = 0$ with support $[-b, +b]$. By differentiating Eq. (7) twice with respect to x, Mossay and Picard [5] derived the spatial equilibrium equation as

$$\frac{\partial^2 \lambda^{(s)}(x)}{\partial x^2} + \delta^2\, \lambda^{(s)}(x) = 0\,, \tag{8}$$

where $\delta^2 = 2\tau/\beta$. Taking into account the continuity of the solution at support edges, the steady-state solution is given by

$$\lambda^{(s)}(x) = \begin{cases} C\cos(\delta x) & \text{for } x \in [-b, +b] \\ 0 & \text{for } x \notin [-b, +b] \end{cases}\,. \tag{9}$$

Now we will be interested mainly in studying solutions that are contained into the invariant manifold $I = 1$. Therefore we can compute the parameters b and the amplitude C as a function of the problem parameters as follows:

$$b = \frac{\pi}{2\delta} = \frac{\pi}{2\sqrt{2}}\sqrt{\frac{\beta}{\tau}}\ ; \quad C = \frac{\delta}{2} = \sqrt{\frac{\tau}{2\beta}}\,. \tag{10}$$

Once the steady-state solution is determined we can compute $\bar{V}$ explicitly as a function of the model parameters:

$$\bar{V} = p = \alpha - \frac{\pi}{2\sqrt{2}}\sqrt{\beta\tau}\,. \tag{11}$$

where we define the parameter p that is linked to the stability of the invariant manifold. If $p > 0$ the invariant manifold $I = 1$ is transversely stable. The solution (9) is represented in Fig. 1 for some specific parameter values.

The above steady state, once the parameters α, β, τ have been fixed, has been shown to be unique, see [5], or [6]. The main purpose of this paper is to study the stability of the