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Inter-occurrence times and universal laws in finance, earthquakes and genomes☆

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ABSTRACT

A plethora of natural, artificial and social systems exist which do not belong to the Boltzmann–Gibbs (BG) statistical-mechanical world, based on the standard additive entropy S_{BG} and its associated exponential BG factor. Frequent behaviors in such complex systems have been shown to be closely related to q-statistics instead, based on the nonadditive entropy S_q (with $S_1 = S_{BG}$), and its associated q-exponential factor which generalizes the usual BG one. In fact, a wide range of phenomena of quite different nature exist which can be described and, in the simplest cases, understood through analytic (and explicit) functions and probability distributions which exhibit some universal features. Universality classes are concomitantly observed which can be characterized through indices such as q. We will exhibit here some such cases, namely concerning the distribution of inter-occurrence (or inter-event) times in the areas of finance, earthquakes and genomes.

1. Historical and physical motivations

In 1865 Clausius introduced in thermodynamics, and named, the concept of *entropy* (noted S, probably in honor of Sadi Carnot, whom Clausius admired) [1]. It was introduced on completely macroscopic terms, with no reference at all to the microscopic world, whose existence was under strong debate at his time, and still even so several decades later. One of the central properties of this concept was to be thermodynamically *extensive*, i.e., to be proportional to the size of the system (characterized by its total mass, for instance). In the 1870s Boltzmann [2,3] made the genius connection of the thermodynamical entropy to the microcosmos. This connection was refined by Gibbs a few years later [4]. From this viewpoint, the thermodynamic extensivity became the nowadays well known property that the total entropy of a system should be proportional to N, the total number of its microscopic elements (or, equivalently, proportional to the total number of microscopic degrees of freedom). More precisely, in the $N \to \infty$ limit, it should asymptotically be

$$S(N) \propto N\,, \tag{1}$$

hence

$$0 < \lim_{N\to\infty} \frac{S(N)}{N} < \infty\,. \tag{2}$$

For a d-dimensional system, $N \propto L^d$, where L is a characteristic linear size and d is either a positive integer number (the standard dimension, basically), or a positive real number (fractal dimension, a concept which was in fact carefully introduced by Hausdorff and fruitfully explored by Mandelbrot). Consequently, Eqs. (1) and (2) can be rewritten as

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follows:

$$S(L) \propto L^d \,, \tag{3}$$

hence

$$0 < \lim_{L\to\infty} \frac{S(L)}{L^d} < \infty \,. \tag{4}$$

The entropic functional introduced by Boltzmann and Gibbs (and later on adapted to quantum and information-theoretical scenarios by von Neumann and Shannon, respectively) is given (for systems described through discrete random variables) by

$$S_{BG}(N) = -k \sum_{i=1}^{W(N)} p_i \ln p_i \quad \left(\sum_{i=1}^{W(N)} p_i = 1 \right), \tag{5}$$

where k is conventional positive constant (usually taken to be Boltzmann constant k_B in physics, and $k = 1$ in several other contexts), and i runs over all non-vanishing-probability microscopic configurations of the N-sized system, $\{p_i\}$ being the corresponding probabilities. In the particular case of equal probabilities, i.e., $p_i = 1/W(N)$ $(\forall i)$, we recover the celebrated Boltzmann formula

$$S_{BG}(N) = k \ln W(N) \,. \tag{6}$$

It is clear that, if the microscopic random variables are probabilistically (strictly or nearly) *independent*, we have

$$W(N) \propto \mu^N \quad (\mu > 1;\ N \to \infty) \,, \tag{7}$$

hence Eq. (6) implies that $S_{BG}(N) \propto N$, thus satisfying the (Clausius) thermodynamic expectation of extensivity, here represented by Eq. (1). If we have N coins (dices), then $\mu = 2$ $(\mu = 6)$; if we have a d-dimensional first-neighbor-interacting Ising ferromagnet in thermal equilibrium with a thermostat, then μ essentially is some temperature-dependent real number.

$W(N)$ might however have a functional dependence drastically different from (7). For example, it could be (see [5–8], and pages 66–68 of [9]; see also [10,11])

$$W(N) \propto N^\rho \quad (\rho > 0;\ N \to \infty) \,, \tag{8}$$

or (see p. 69 of [9])

$$W(N) \propto \nu^{N^\gamma} \quad (\nu > 1;\ 0 < \gamma < 1;\ N \to \infty) \,. \tag{9}$$

Such cases[1] clearly correspond to probabilistically *strong* correlations, of different nature though. We easily verify that, for $N \to \infty$,

$$1 \ll N^\rho \ll \nu^{N^\gamma} \ll \mu^N \,. \tag{10}$$

This is directly related to strong restrictions which mandate an occupancy of the entire phase space *substantially lesser than full (or nearly full) occupancy* (which corresponds in turn to Eq. (7), and, for nonlinear dynamical systems, to ergodicity). We may say alternatively that Eq. (7) is to be associated with an occupancy of phase space with *finite* Lebesgue measure, whereas Eqs. (8) and (9) typically correspond to an occupancy with *zero* Lebesgue measure.[2]

If we assume – and we do, for reasons to be presented hereafter – that entropic extensivity (i.e., Eq. (1)) must hold *in all cases*, we are forced to generically abandon the BG functional (5) whenever probabilistically strong correlations are generically present in the system. This is the primary physical and mathematical origin of the nonadditive entropies introduced in [5] in order to generalize the BG entropy S_{BG} and also concomitantly generalize the BG statistical mechanics. This is fully consistent with crucial remarks by Boltzmann, Gibbs, Fermi, Majorana, Tisza, Landsberg, and various others (see, for instance, Chapter 1 of [9]) pointing the limits of validity of the BG basic hypothesis. In the next section we show how *nonadditive* entropic functionals (e.g., S_q introduced in [5] in order to generalize the BG theory) become mandatory in order to satisfy this demand in those cases which overcome the usual BG frame and its *additive* functional S_{BG}.

2. Thermodynamical entropic extensivity in strongly correlated systems generically mandates nonadditive entropic functionals

In what follows we shall refer to *uncorrelated* or *weakly correlated* N-body systems whenever Eq. (7) occurs, and to *strongly correlated* ones whenever zero-Lebesgue-measure behaviors such as those in Eqs. (8) and (9) occur.

Let us introduce now the following entropic functional $(q \in \mathcal{R})$:

$$S_q = k \frac{1 - \sum_{i=1}^{W} p_i^q}{q-1} \quad (S_1 = S_{BG}) \,. \tag{11}$$

This expression can be equivalently rewritten as follows:

$$S_q = k \sum_{i=1}^{W} p_i \ln_q \frac{1}{p_i} = -k \sum_{i=1}^{W} p_i^q \ln_q p_i = -k \sum_{i=1}^{W} p_i \ln_{2-q} p_i, \tag{12}$$

where

$$\ln_q z \equiv \frac{z^{1-q} - 1}{1-q} \quad (\ln_1 z = \ln z) \,. \tag{13}$$

[1] Usually W increases with N, but it is not forbidden that it asymptotically decreases with N (see [12] for instance). Therefore, it might in principle also occur $\mu < 1$ in Eq. (7), $\rho < 0$ in Eq. (8), and $\nu < 1$ with $\gamma > 0$ in Eq. (9). Of course, in such pathological cases, an additive constant of the order of unity must be included in the asymptotic behaviors in order to never violate $W \geq 1$.

[2] Let us further analyze this case. If we have N distinguishable particles, each of them living in a continuous D-dimensional space ($D = 2d$ if the system is defined in terms of canonically conjugate dynamical variables of a d-dimensional system; for example, Gibbs Γ phase space is a $2dN$-dimensional space), then the full space of possibilities is a DN-dimensional hypercube whose hypervolume equals D^N (under the assumption that virtually all these possibilities have nonzero probability to occur). In such a case, its Lebesgue measure scales precisely as $W(N) \sim D^N$, in conformity with Eq. (7) with $\mu = D$. Strong correlations in such a system *cannot increase* its Lebesgue measure, but can of course decrease it, and even make it to be *zero*, as are the cases corresponding to Eqs. (8) and (9). However, in remarkable contrast with the standard situation represented by Eq. (7), systems do exist whose total number of possibilities can increase even faster than μ^N. Such is the case of N ranked elements. Indeed, the amount of all possible rankings yields $W(N) = N!$ [13]. Consequently the BG entropy given by Eq. (5) yields $S_{BG}(N) \sim kN \ln N$, which does not conform to thermodynamics. What precise entropic form would recover extensivity for such a case is at present an interesting open question.