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A note on bi-periodic Fibonacci and Lucas quaternions

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ABSTRACT

Motivated by the our recent work in Tan et al., 2016, related to the bi-periodic Fibonacci quaternions, here we introduce the bi-periodic Lucas quaternions that gives the Lucas quaternions as a special case. We give the generating function and the Binet formula for these quaternions. Also, we give the relationships between bi-periodic Fibonacci quaternions and bi-periodic Lucas quaternions.

1. Introduction

A quaternion is a hyper-complex number and defined by

$$q = q_0 + q_1 i_1 + q_2 i_2 + q_3 i_3,$$

where q_0, q_1, q_2, q_3 are real numbers and i_1, i_2, i_3 are standard orthonormal basis in $\mathbb{R}^3$. The quaternion multiplication satisfy the following rules:

$$i_1^2 = i_2^2 = i_3^2 = -1,$$
$$i_1 i_2 = i_3 = -i_2 i_1, \; i_2 i_3 = i_1 = -i_3 i_2, \; i_3 i_1 = i_2 = -i_1 i_3. \quad (1.1)$$

The conjugate of the quaternion q is denoted by $\overline{q}$ and defined by

$$\overline{q} = q_0 - q_1 i_1 - q_2 i_2 - q_3 i_3$$

where i_1, i_2 and i_3 satisfies the multiplication rules (1.1). It is well known that the quaternions are members of noncommutative algebra. There has been an increasing interest on quaternions that play an important role in various areas such as computer sciences, physics, differential geometry, quantum physics, signal, color image processing, geostatics and analysis. For a survey on quaternions we refer to [1,5,8].

The Fibonacci numbers F_n are defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}, \; n \geq 2$$

with the initial conditions $F_0 = 0$ and $F_1 = 1$. The Lucas numbers L_n, which follows the same recursive pattern as the Fibonacci numbers, but begins with $L_0 = 2$ and $L_1 = 1$. These numbers are famous for possessing wonderful properties and important applications to diverse disciplines such as mathematics and computer science.

There are several researches on different types of sequences of quaternions. Horadam [9] defined the Fibonacci and the Lucas quaternions as:

$$Q_n = F_n + F_{n+1} i_1 + F_{n+2} i_2 + F_{n+3} i_3$$

and

$$P_n = L_n + L_{n+1} i_1 + L_{n+2} i_2 + L_{n+3} i_3$$

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respectively, where F_n is the nth Fibonacci number and L_n is the nth Lucas number. In [17], Ramirez defined the k-Fibonacci and k-Lucas quaternions as:

$$D_{k,n} = F_{k,n} + F_{k,n+1}i_1 + F_{k,n+2}i_2 + F_{k,n+3}i_3$$

and

$$P_{k,n} = L_{k,n} + L_{k,n+1}i_1 + L_{k,n+2}i_2 + L_{k,n+3}i_3$$

respectively, where $F_{k,\,n}$ is the nth k-Fibonacci number and $L_{k,\,n}$ is the nth k-Lucas number. Some results on Fibonacci and Lucas quaternions can be found in [6,7,10–16,18].

Recently, Tan et al. [20] introduced a new generalization of the Fibonacci quaternions, named as, the bi-periodic Fibonacci quaternions Q_n as:

$$Q_n = \sum_{l=0}^{3} q_{n+l}e_l, \;\; n \geq 0 \tag{1.2}$$

where q_n is the n-th bi-periodic Fibonacci number and defined by

$$q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{if } n \text{ is even} \\ bq_{n-1} + q_{n-2}, & \text{if } n \text{ is odd} \end{cases}, \;\; n \geq 2$$

with initial values $q_0 = 0, \; q_1 = 1$ and $a, \, b$ are nonzero numbers (see [21]). They are emerged as a generalization of the best known quaternions in the literature, such as classical Fibonacci quaternions, Pell quaternions, k-Fibonacci quaternions. The generating function, the Binet formula, and several identities of the bi-periodic Fibonacci quaternions were obtained. Also, the authors corrected the results in [3] and [4] which have been overlooked that the quaternion multiplication is noncommutative.

Motivated by the our recent work in [20], related to the bi-periodic Fibonacci quaternions, here we introduce the bi-periodic Lucas quaternions that gives the Lucas quaternions as a special case. We give the generating function and the Binet formula for these quaternions. By using the Binet formula we obtain several identities for these quaternions. One of the main importance of this paper is to state the relationships between bi-periodic Fibonacci quaternions and bi-periodic Lucas quaternions.

The outline of this paper is as follows: In the rest of this section, we introduce some necessary definitions and mathematical preliminaries, which is required; in Section 2, we introduce the bi-periodic Lucas quaternions and give the generating function, the Binet formula, and some identities for these quaternions.

We start by recalling some basic results concerning quaternion algebra $\mathbf{H}$, bi-periodic Fibonacci quaternions, and bi-periodic Lucas numbers.

It is well known that the algebra $\mathbf{H} = \{a = a_0e_0 + a_1e_1 + a_2e_2 + a_3e_3 : a_i \in \mathbb{R}, \; i = 0, 1, 2, 3\} \cong \mathbb{C}^2$ of real quaternions is defined as the four-dimensional vector space over $\mathbb{R}$ having a basis $e_0 \cong 1$, $e_1 \cong i_1$, $e_2 \cong i_2$ and $e_3 \cong i_3$, which satisfies the following multiplication rules:

$$e_l^2 = -1, \;\; l \in \{1, 2, 3\};$$
$$e_1e_2 = -e_2e_1 = e_3, \;\; e_2e_3 = -e_3e_2 = e_1, \;\; e_3e_1 = -e_1e_3 = e_2. \tag{1.3}$$

The generating function for the bi-periodic Fibonacci quaternions Q_n is

$$F(t) = \frac{Q_0 + (Q_1 - bQ_0)t + (a-b)R(t)}{1 - bt - t^2} \tag{1.4}$$

where

$$R(t) := tf(t)e_0 + (f(t) - t)e_1 + \left(\frac{f(t)}{t} - 1\right)e_2 + \left(\frac{f(t) - \left(t + (ab+1)t^3\right)}{t^2}\right)e_3,$$

$$f(t) := \sum_{n=1}^{\infty} q_{2n-1}t^{2n-1} = \frac{t - t^3}{1 - (ab+2)t^2 + t^4}.$$

Also, the Binet formula for the bi-periodic Fibonacci quaternion is given by

$$Q_n = \begin{cases} \dfrac{1}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor}} \dfrac{\alpha^*\alpha^n - \beta^*\beta^n}{\alpha - \beta}, & \text{if } n \text{ is even} \\ \dfrac{1}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor}} \dfrac{\alpha^{**}\alpha^n - \beta^{**}\beta^n}{\alpha - \beta}, & \text{if } n \text{ is odd} \end{cases} \tag{1.5}$$

where

$$\alpha^* := \sum_{l=0}^{3} \frac{a^{\xi(l+1)}}{(ab)^{\left\lfloor \frac{l}{2} \right\rfloor}} \alpha^l e_l, \;\; \beta^* := \sum_{l=0}^{3} \frac{a^{\xi(l+1)}}{(ab)^{\left\lfloor \frac{l}{2} \right\rfloor}} \beta^l e_l \tag{1.6}$$

$$\alpha^{**} := \sum_{l=0}^{3} \frac{a^{\xi(l)}}{(ab)^{\left\lfloor \frac{l+1}{2} \right\rfloor}} \alpha^l e_l, \;\; \beta^{**} := \sum_{l=0}^{3} \frac{a^{\xi(l)}}{(ab)^{\left\lfloor \frac{l+1}{2} \right\rfloor}} \beta^l e_l. \tag{1.7}$$

Bilgici [2] introduced the bi-periodic Lucas numbers as:

$$p_n = \begin{cases} bp_{n-1} + p_{n-2}, & \text{if } n \text{ is even} \\ ap_{n-1} + p_{n-2}, & \text{if } n \text{ is odd} \end{cases}, \;\; n \geq 2 \tag{1.8}$$

with the initial conditions $p_0 = 2$ and $p_1 = a$. It should also be noted that, it gives the Lucas sequence in the case of $a = b = 1$ in $\{p_n\}$ we obtain the classical Lucas sequence, if we take $a = b = 2$ in $\{p_n\}$, we get the Pell–Lucas sequence, and if we take $a = b = k$ in $\{p_n\}$, we get the k-Lucas sequence. The bi-periodic Lucas numbers satisfy the following recurrence;

$$p_n = (ab+2)p_{n-2} - p_{n-4}, \;\; n \geq 4.$$

The generating function of the sequence $\{p_n\}$ is

$$L(x) = \frac{2 + ax - (ab+2)x^2 + ax^3}{1 - (ab+2)x^2 + x^4}$$

and the Binet formula of the sequence $\{p_n\}$ is given by

$$p_n = \frac{a^{\xi(n)}}{(ab)^{\left\lfloor \frac{n+1}{2} \right\rfloor}} (\alpha^n + \beta^n) \tag{1.9}$$

where $\alpha = \frac{ab + \sqrt{a^2b^2 + 4ab}}{2}$ and $\beta = \frac{ab - \sqrt{a^2b^2 + 4ab}}{2}$ that is, α and β are the roots of the polynomial $x^2 - abx - ab$ and $\xi(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor$, i.e., $\xi(n) = 0$ when n is even and $\xi(n) = 1$ when n is odd.

For further information about the bi-periodic Fibonacci quaternions and the bi-periodic Lucas numbers see [2,19,20].