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LCK rank of locally conformally Kähler manifolds with potential

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ABSTRACT

An LCK manifold with potential is a quotient of a Kähler manifold X equipped with a positive Kähler potential f, such that the monodromy group acts on X by holomorphic homotheties and multiplies f by a character. The LCK rank is the rank of the image of this character, considered as a function from the monodromy group to real numbers. We prove that an LCK manifold with potential can have any rank between 1 and $b_1(M)$. Moreover, LCK manifolds with proper potential (ones with rank 1) are dense. Two *errata* to our previous work are given in the last section.

1. Introduction

1.1. LCK manifolds

A complex manifold (M, I) is called **locally conformally Kähler** (LCK) if it admits a Hermitian metric g and a closed 1-form θ, called **the Lee form**, such that the fundamental 2-form $\omega(\cdot,\cdot) := g(\cdot, I\cdot)$ satisfies the integrability condition

$$d\omega = \theta \wedge \omega, \quad d\theta = 0. \tag{1.1}$$

The above definition is equivalent (see [1]) to the existence of a covering $\tilde{M}$ endowed with a Kähler metric Ω which is acted on by the deck group $\mathrm{Aut}_M(\tilde{M})$ by holomorphic homotheties. Hence, if $\tau \in \mathrm{Aut}_M(\tilde{M})$, then $\tau^*\Omega = c_\tau \cdot \Omega$, where $c_\tau \in \mathbb{R}^{>0}$ is the scale factor. This defines a character

$$\chi : \mathrm{Aut}_M(\tilde{M}) \longrightarrow \mathbb{R}^{>0}, \quad \chi(\tau) = c_\tau. \tag{1.2}$$

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Two subclasses of LCK manifolds will be of interest to us.

The **Vaisman** class is formed by LCK manifold (M, ω, θ) with parallel Lee form with respect to the Levi-Civita connection of g. While the LCK condition is conformally invariant (if g is LCK, then any $e^f \cdot g$ is still LCK, with Lee form $\theta + df$), the Vaisman condition is not. The main example of Vaisman manifold is the diagonal Hopf manifold [2]. Also, all compact complex submanifolds of Vaisman manifolds are Vaisman, too, [3]. The Vaisman compact complex surfaces are classified in [4].

We observed in [3,5] that the Kähler form of the universal cover of any Vaisman manifold has global potential represented by the square of the length of the Lee form. Moreover, the deck group acts on the potential by multiplying it with the character χ. This led us to introducing the larger class of LCK manifolds **with potential**. The precise definition requires the existence of a Kähler covering on which the Kähler metric has global, positive and proper potential function which is acted on by homotheties by the deck group. Besides Vaisman manifolds, there exist non-Vaisman examples, such as the non-diagonal Hopf manifolds, [5].

1.2. LCK manifolds with potential

"LCK manifolds with (proper) potential" can be defined as LCK manifolds (M, ω, θ) equipped with a smooth function $\psi \in \mathcal{C}^\infty(M)$ such that

$$\omega = d_\theta d_\theta^c \psi, \tag{1.3}$$

where $d_\theta(x) = dx - \theta \wedge x$, $d_\theta^c = Id_\theta I^{-1}$, and the following properties are satisfied:

$$\begin{aligned}&\text{(i)}\ \ \psi > 0;\\ &\text{(ii)}\ \ \text{the class } [\theta] \in H^1(M, \mathbb{R}) \text{ is proportional to a rational one.}\end{aligned} \tag{1.4}$$

For more details and historical context of this definition, please see Section 2.1. The differential d_θ is identified with the de Rham differential with coefficients in a flat line bundle L called **the weight bundle**. In this context, ψ should be considered as a section of L. After passing to the smallest covering $\tilde{M} \xrightarrow{\pi} M$ where θ becomes exact, the pull-back bundle π^*L can be trivialized by a parallel section. Then the equation (1.3) becomes $\tilde{\omega} = dd^c\tilde{\psi}$, where $\tilde{\omega}$ is a Kähler form on $\tilde{M}$, and $\tilde{\psi}$ the Kähler potential.

Since $\tilde{M} \xrightarrow{\pi} M$ is the smallest covering where θ becomes exact, its monodromy is equal to $\mathbb{Z}^k$, where k is the rank of the smallest rational subspace $V \subset H^1(M, \mathbb{Q})$ such that $V \otimes_{\mathbb{Q}} \mathbb{R}$ contain $[\theta]$. In particular, the condition (1.4)(ii) means precisely that $\tilde{M} \xrightarrow{\pi} M$ is a $\mathbb{Z}$-covering. This implies that the definition (1.3)–(1.4) is equivalent to the historical one (Definition 2.1).

However, the condition (1.4)(ii) is more complicated: there are examples of LCK manifolds satisfying (1.3) and not (1.4)(ii) (Section 2.3). Still, any complex manifold (M, ω, θ) admitting an LCK metric with potential ψ satisfying (1.3), admits an LCK metric satisfying (1.3)–(1.4) in any $\mathcal{C}^\infty$-neighbourhood of (ω, θ). Therefore the condition (1.4)(ii) is not restrictive, and for most applications, unnecessary.

It makes sense to modify the notion of LCK manifold with potential to include the following notion (Section 2.3):

Definition 1.1. Let (M, ω, θ) be an LCK manifold, and $\psi \in \mathcal{C}^\infty(M)$ a strictly positive function satisfying $d_\theta d_\theta^c \psi = \omega$. Denote by k the rank of the smallest rational subspace $V \subset H^1(M, \mathbb{Q})$ such that $V \otimes_{\mathbb{Q}} \mathbb{R}$ contain $[\theta]$. Then ψ is called **proper potential** if $k = 1$ and **improper potential** if $k > 1$.

1.3. Some errors found

This paper is much influenced by Paul Gauduchon, who discovered an error in our result mentioned as obvious in [6]. In [6], we claimed erroneously that an LCK metric is pluricanonical (*i.e.* $(\nabla\theta)^{1,1} = 0$, see [7]), if and only if it admits an LCK potential. This was obvious because (as we claimed) the equations for LCK with potential and for pluricanonical metric are the same. Unfortunately, a scalar multiplier was missing in our equation for the pluricanonical metric.

From an attempt to understand what is brought by the missing multiplier, this paper grew, and we found an even stronger result: any compact pluricanonical manifold is Vaisman. Very recently, Andrei and Sergiu Moroianu gave a simple, direct proof of this result, using elegant tensor computations, [8].

However, during our work trying to plug a seemingly harmless mistake, we discovered a much more offensive error, which has proliferated in a number of our papers.

In [9], we claimed that any Vaisman manifold admits a $\mathbb{Z}$-covering which is Kähler. This is true for locally conformally hyperkähler manifolds, as shown in [3]. However, this result is false for more general Vaisman manifolds, such as a Kodaira surface (Theorem 3.4).

It is easiest to state this problem and its solution using the notion of "LCK rank" (Definition 2.5), defined in [10] and studied in [11]. Briefly, the LCK rank is the smallest r such that there exists a $\mathbb{Z}^r$-covering $\tilde{M}$ of M such that the pullback of the LCK metric is conformally equivalent to a Kähler metric on $\tilde{M}$.