Journal of Geometry and Physics 108 (2016) 71–82

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Low-dimensional filiform Lie superalgebras

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ARTICLE INFO

Article history:
Received 28 March 2016
Received in revised form 18 May 2016
Accepted 21 June 2016
Available online 2 July 2016

MSC:
17B30
17B56

Keywords:
Lie algebras
Lie superalgebras
Cohomology
Deformation
Nilpotent
Filiform

ABSTRACT

The present work is regarding *filiform Lie superalgebras* which is an important type of nilpotent Lie superalgebras. In general, classifying nilpotent Lie superalgebras is at present an open and unsolved problem. Throughout the present work we contribute to the resolution of this wide problem by classifying filiform Lie superalgebras of low dimensions, in particular less or equal to 7. Furthermore we would establish a method that could be applied to obtain similar results for higher dimensions. Thus, this method would mainly consist in using infinitesimal deformations of the model filiform Lie superalgebra.

1. Introduction

The concept of *filiform Lie algebras* was firstly introduced in [1] by Vergne. Moreover this type of nilpotent Lie algebras has important properties such as every filiform Lie algebra can be obtained by a deformation of the model filiform algebra L_n.

The present work is regarding *filiform Lie superalgebras*, a generalization of filiform Lie algebras and an important type of nilpotent Lie superalgebras. It has been proved that in the same way as filiform Lie algebras, all filiform Lie superalgebras can be obtained by infinitesimal deformations of the model Lie superalgebra $L^{n,m}$. Thus, throughout this work we give a complete classification (up to isomorphisms) of complex filiform Lie superalgebras of dimension less or equal to 7 by means of the above mentioned deformations. Furthermore our method could be used in higher dimensions.

We will therefore consider infinitesimal deformations of $L^{n,m}$ which are defined by even 2-cocycles in $Z_0^2(L^{n,m}, L^{n,m})$. These deformations have been almost totally determined along the papers [2–5]. In some particular cases before applying our method we had to obtain some of these deformations according to the instructions described in the above mentioned papers.

All the vector spaces that appear in this paper (and thus, all the algebras) are assumed to be $\mathbb{C}$-vector spaces of finite dimension. Moreover, we shall use the well-known convention that for the definition of a (super) Lie bracket in terms of a basis only the non-vanishing brackets in some ordering of the base are explicitly mentioned.

2. Preliminaries

A *superspace* is nothing but a vector space with a $\mathbb{Z}_2$-grading: $V = V_0 \oplus V_1$. Elements of the space V_0 are usually called even, and elements of the space V_1, odd; the indices 0 and 1 are modulo 2. A linear map $\phi : V \to W$ between two super

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http://dx.doi.org/10.1016/j.geomphys.2016.06.012
0393-0440/

vector spaces is called *even* iff $\phi(V_0) \subset W_0$ and $\phi(V_1) \subset W_1$ and is called *odd* iff $\phi(V_0) \subset W_1$ and $\phi(V_1) \subset W_0$. Thus, we will have $\mathrm{Hom}(V,W) = \mathrm{Hom}(V,W)_0 \oplus \mathrm{Hom}(V,W)_1$ where the first summand is composed by all the even and the second summand by all the odd linear maps.

A *Lie superalgebra* (see [6,7]) is a superspace $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, with an even bilinear commutation operation (or "supercommutation") [,], which satisfies the conditions:

1. $[X,Y] = -(-1)^{\alpha\cdot\beta}[Y,X] \quad \forall X \in \mathfrak{g}_\alpha, \forall Y \in \mathfrak{g}_\beta$.
2. $(-1)^{\gamma\cdot\alpha}[X,[Y,Z]] + (-1)^{\alpha\cdot\beta}[Y,[Z,X]] + (-1)^{\beta\cdot\gamma}[Z,[X,Y]] = 0$

for all $X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_\beta, Z \in \mathfrak{g}_\gamma$ with $\alpha, \beta, \gamma \in \mathbb{Z}_2$. (Graded Jacobi identity).

Thus, $\mathfrak{g}_0$ is an ordinary Lie algebra, and $\mathfrak{g}_1$ is a module over $\mathfrak{g}_0$; the Lie superalgebra structure also contains the symmetric pairing $S^2\mathfrak{g}_1 \longrightarrow \mathfrak{g}_0$.

The *descending central sequence* of a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is defined by $\mathcal{C}^0(\mathfrak{g}) = \mathfrak{g}$, $\mathcal{C}^{k+1}(\mathfrak{g}) = [\mathcal{C}^k(\mathfrak{g}), \mathfrak{g}]$ for all $k \geq 0$. If $\mathcal{C}^k(\mathfrak{g}) = \{0\}$ for some k, the Lie superalgebra is called *nilpotent*. The smallest integer k such as $\mathcal{C}^k(\mathfrak{g}) = \{0\}$ is called the *nilindex* of $\mathfrak{g}$. Analogously as for Lie algebras, the dimension of each term of the descending central sequence is an invariant of the superalgebra. Likewise it can be noted that each term of this sequence has an even part and an odd part, that is $\mathcal{C}^k(\mathfrak{g}) = (\mathcal{C}^k(\mathfrak{g}))_0 \oplus (\mathcal{C}^k(\mathfrak{g}))_1$, whose dimensions are also invariants of the superalgebra.

There are also defined two other descending sequences called $\mathcal{C}^k(\mathfrak{g}_0)$ and $\mathcal{C}^k(\mathfrak{g}_1)$:

$$\mathcal{C}^0(\mathfrak{g}_i) = \mathfrak{g}_i, \qquad \mathcal{C}^{k+1}(\mathfrak{g}_i) = [\mathfrak{g}_0, \mathcal{C}^k(\mathfrak{g}_i)], \quad k \geq 0, i \in \{0,1\}.$$

If $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a nilpotent Lie superalgebra, then $\mathfrak{g}$ has super-nilindex or *s-nilindex* (p,q), if the following conditions hold:

$$(\mathcal{C}^{p-1}(\mathfrak{g}_0)) \neq 0 \qquad (\mathcal{C}^{q-1}(\mathfrak{g}_1)) \neq 0, \qquad \mathcal{C}^p(\mathfrak{g}_0) = \mathcal{C}^q(\mathfrak{g}_1) = 0.$$

It can be noted that a *module* $A = A_0 \oplus A_1$ of the Lie superalgebra $\mathfrak{g}$ is an even bilinear map $\mathfrak{g} \times A \to A$ satisfying

$$\forall X \in \mathfrak{g}_\alpha, \qquad Y \in \mathfrak{g}_\beta\ a \in A: \qquad X(Ya) - (-1)^{\alpha\beta}Y(Xa) = [X,Y]a.$$

Lie superalgebra cohomology is defined in the following well-known way (see e.g. [6,8]): the superspace of *q-dimensional cocycles* of the Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with coefficients in the $\mathfrak{g}$-module $A = A_0 \oplus A_1$ is given by

$$C^q(\mathfrak{g}; A) = \bigoplus_{q_0+q_1=q} \mathrm{Hom}\left(\wedge^{q_0}\mathfrak{g}_0 \otimes S^{q_1}\mathfrak{g}_1, A\right).$$

The above space is graded by $C^q(\mathfrak{g};A) = C^q_0(\mathfrak{g};A) \oplus C^q_1(\mathfrak{g};A)$ with

$$C^q_p(\mathfrak{g};A) = \bigoplus_{\substack{q_0+q_1=q \\ q_1+r\equiv p \bmod 2}} \mathrm{Hom}\left(\wedge^{q_0}\mathfrak{g}_0 \otimes S^{q_1}\mathfrak{g}_1, A_r\right).$$

Thus we have the *cohomology groups*

$$H^q_p(\mathfrak{g};A) = Z^q_p(\mathfrak{g};A)/B^q_p(\mathfrak{g};A)$$

where, in particular, the elements of $Z^q_0(\mathfrak{g};A)$ and $Z^q_1(\mathfrak{g};A)$ are called *even q-cocycles* and *odd q-cocycles* respectively.

On the other hand and in complete analogy to Lie algebras [9–11] we denote by $\mathcal{N}^{n+1,m}$ the variety of nilpotent Lie superalgebras $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with $\dim \mathfrak{g}_0 = n+1$ and $\dim \mathfrak{g}_1 = m$. Thus, we will have the following definition

Definition 2.1 (*[12]*). Any nilpotent Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \in \mathcal{N}^{n+1,m}$ with s-nilindex (n,m) is called *filiform*.

We denote by $\mathcal{F}^{n+1,m}$ the subset of $\mathcal{N}^{n+1,m}$ consisting of all the filiform Lie superalgebras.

Before studying this family of Lie superalgebras it is convenient to solve the problem of finding a suitable basis, a so-called *adapted basis*.

Theorem 2.1 ([12]). *If $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \in \mathcal{F}^{n+1,m}$, then there exists an adapted basis of $\mathfrak{g}$, namely $\{X_0, X_1, \ldots, X_n, Y_1, \ldots, Y_m\}$, with $\{X_0, X_1, \ldots, X_n\}$ a basis of $\mathfrak{g}_0$ and $\{Y_1, \ldots, Y_m\}$ a basis of $\mathfrak{g}_1$, such that:*

$$\begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leq i \leq n-1, \\ [X_0, X_n] = 0, & \\ [X_0, Y_j] = Y_{j+1}, & 1 \leq j \leq m-1, \\ [X_0, Y_m] = 0. & \end{cases}$$

X_0 is called the characteristic vector.

From now on all the filiform superalgebras, that we are going to consider throughout the present paper, would be expressed in an adapted basis.