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Natural spinor structures over Lorentzian manifolds

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ABSTRACT

We show that any 4-dimensional Lorentzian manifold with nondegenerate Weyl tensor possesses the spinor structure and this structure is natural, i.e. invariant under action of diffeomorphisms.

1. Introduction

The main postulate of the relativity theory requires that differential equations guiding theory should be natural, i.e., independent of lab, or invariant with respect to the group of diffeomorphisms. This is obviously true for the Einstein and the Einstein–Maxwell equations, but adding other physical fields to the theory meets the visual obstruction. Namely, when we talk on naturality of equations we should provide naturality of the physical fields involved in consideration. In other words, one should require that the fields are sections of the natural bundles. It is correct for gravitational and electromagnetic fields, but not for arbitrary fields.

In this paper we show that any 4-dimensional Lorentzian manifold (M, g), with nondegenerate (in the sense of the Petrov classification [1]) Weyl tensor, possesses a natural spinor structure. Therefore, the Einstein–Dirac and the Einstein–Maxwell–Dirac equations could be written in a natural way.

This paper deals with the spinor structure only. The application to the Einstein–Dirac and the Einstein–Maxwell–Dirac equations as well as investigation of the corresponding differential invariants (similar to [2]) shall be done in the next papers.

The paper is organized as follows.

Section 2 contains necessary facts about operators in exterior algebra $\Lambda^{\cdot} E^*$ of the Minkowski space (E, g).

In Section 3, we investigate a module structure over the Clifford algebra $\mathrm{Cl}(E, g)$ for the exterior algebra $\Lambda^{\cdot} E^*$.

In Section 4, we recall main properties of the Weyl operator of a Lorentzian metric g of signature $(1, 3)$. We show that eigenvectors of the nondegenerate Weyl operator (Petrov type **I**) generate a natural family of orthonormal bases ("canonical"

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bases) at every point $a \in M$ of an oriented and time-oriented Lorentzian manifold (M, g). These bases are defined up to the action of the group $\mathbb{G} = \mathbb{Z}_3 \times \mathbb{Z}_2^2$.

In Section 5, we show that nondegenerate Weyl operator generates the direct sum decomposition $\Lambda^{\cdot} T_a^* = \mathbb{S}_+(g) \oplus \mathbb{S}_-(g)$, where $\mathbb{S}_\pm(g)$ are irreducible $\mathrm{Cl}(T_a, g)$-modules, at every point $a \in M$ of the oriented and time-oriented Lorentzian manifold (M, g). Elements of these modules are spinors.

In addition, we show that every "canonical" orthonormal base generates an isomorphism $\mathrm{Cl}(T_a, g) \simeq \mathrm{Mat}_2(\mathbb{H})$, where $\mathbb{H}$ is the quaternion algebra, and these isomorphisms are conjugated by the action of the group $\mathbb{G}$.

At last in Section 6, we show that

1. there exists the natural direct sum decomposition
$$\Lambda^{\cdot} T^* = \mathbb{S}_+(g) \oplus \mathbb{S}_-(g),$$
where $\mathbb{S}_\pm(g)$ are irreducible $\mathrm{Cl}\big(T, g\big)$-submodules;
2. moreover, if a fundamental group $\pi(M)$ does not contain elements of order 2 and 3, then the spinor bundles $\mathbb{S}_\pm(g)$ are trivial and the isomorphism $\mathrm{Cl}\big(T, g\big) \simeq \mathrm{Mat}_2(\mathbb{H})$ is valid.

2. Exterior algebra over Minkowski vector space

Let (E, g) be a 4-dimensional Minkowski vector space equipped with $(1, 3)$-metric g. This metric induces the metric in the dual vector space E^* as well as the metrics in the tensor algebra over E. To simplify notions, we shall use the same letter g for all these metrics.

We also assume, that the vector space E is oriented.

By $\Omega_g \in \Lambda^4 E^*$ we denote the volume form generated by the metric g, $g(\Omega_g, \Omega_g) = -1$.

Metric g induces isomorphism $\widehat{g} : E \to E^*$, where $\big(\widehat{g}(X)\big)(Y) = g(X, Y)$ for all $X, Y \in E$.

We identify vectors and covectors by means of this isomorphism, and denote by $\widehat{\theta} \in E$ the vector corresponding to covector $\theta \in E^*$, $\widehat{g}(\widehat{\theta}) = \theta$.

Let
$$\Lambda^{\cdot} E^* = \oplus_{k=0}^4 \Lambda^k E^*$$
be the exterior algebra.

Then any covector $\theta \in E^*$ generates two operators:

exterior multiplication
$$e_\theta : \Lambda^k E^* \to \Lambda^{k+1} E^*,$$
$$e_\theta : \omega \mapsto \theta \wedge \omega,$$
and inner derivation
$$i_\theta : \Lambda^k E^* \to \Lambda^{k-1} E^*,$$
$$i_\theta : \omega \mapsto \widehat{\theta} \rfloor \omega.$$

These operators satisfy to the following well known commutation relations:
$$\begin{aligned} i_{\theta_1} \circ i_{\theta_2} + i_{\theta_2} \circ i_{\theta_1} &= 0, \\ e_{\theta_1} \circ e_{\theta_2} + e_{\theta_2} \circ e_{\theta_1} &= 0, \\ i_{\theta_1} \circ e_{\theta_2} + e_{\theta_2} \circ i_{\theta_1} &= g(\theta_1, \theta_2) \end{aligned}$$
for all covectors $\theta_1, \theta_2 \in E^*$.

Denote by
$$* : \Lambda^k E^* \to \Lambda^{4-k} E^*$$
the Hodge $*$-operator, where
$$*(\theta_1 \wedge \cdots \wedge \theta_k) = i_{\theta_k} \circ \cdots \circ i_{\theta_1}(\Omega_g).$$
Then, on the Minkowski space, this operator satisfies the following properties:
$$\begin{aligned} &*^2 = -1, \\ &*(\Omega_g) = -1, \qquad *(1) = \Omega_g, \\ &* \circ e_\theta = i_\theta \circ * \end{aligned}$$
for all $\theta \in E^*$.