Journal of Geometry and Physics 106 (2016) 102–107

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Representations of the affine-Virasoro algebra of type A_1

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ARTICLE INFO

Article history:
Received 14 March 2015
Received in revised form 29 September 2015
Accepted 4 March 2016
Available online 17 March 2016

MSC:
primary 17B68
secondary 17B67

Keywords:
Virasoro algebra
Affine-Virasoro algebras
Weight module

ABSTRACT

In this paper, we classify all irreducible weight modules over the affine-Virasoro Lie algebra of type A_1 with finite dimensional weight spaces.

1. Introduction

It is well known that the affine Lie algebras and the Virasoro algebra have been widely used in many physics areas and mathematical branches, and the Virasoro algebra served as an outer-derivative subalgebra plays a key role in representation theory of the affine Lie algebras. Their close relationship strongly suggests that they should be considered simultaneously, i.e., as one algebraic structure. Actually it has led to the definition of the so-called affine-Virasoro algebra [1,2], which is the semidirect product of the Virasoro algebra and an affine Kac–Moody Lie algebra with a common center. Affine-Virasoro algebras sometimes are much more connected to the conformal field theory. For example, the even part of the $N = 3$ superconformal algebra is just the affine-Virasoro algebra of type A_1. Highest weight representations and integrable representations of the affine-Virasoro algebras have been studied in several papers (see [1,3,2,4–8], etc.). However, up to now, all irreducible weight modules with finite dimensional weight spaces (also named Harish-Chandra modules) over these algebras are not yet classified.

In this paper, we classify all irreducible weight modules over the affine-Virasoro Lie algebra of type A_1 with finite dimensional weight spaces. Throughout this paper, $\mathbb{Z}$, $\mathbb{Z}^*$ and $\mathbb{C}$ denote the sets of integers, non-zero integers and complex numbers, respectively. All modules considered in this paper are nontrivial. Let $U(L)$ denote the universal enveloping algebra for a Lie algebra L. For any $\mathbb{Z}$-graded space G, we also use notations G_+, G_-, G_0 and $G_{[p,q)}$ to denote the subspaces spanned by elements in G of degree k with $k > 0$, $k < 0$, $k = 0$ and $p \leq k < q$, respectively.

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http://dx.doi.org/10.1016/j.geomphys.2016.03.009
0393-0440/

2. Basics

In this section, we shall introduce some notations of the Virasoro algebra and affine-Virasoro algebras.

2.1. Virasoro algebra and twisted Heisenberg–Virasoro algebra

By definition, the Virasoro algebra Vir $:= \mathbb{C}\{d_m, C \mid m \in \mathbb{Z}\}$ with bracket:

$$[d_m, d_n] = (n-m)d_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}C, \qquad [d_m, C] = 0, \tag{2.1}$$

for all $m, n \in \mathbb{Z}$.

Let $\mathbb{C}[t, t^{-1}]$ be the Laurent polynomials ring over $\mathbb{C}$, then $\operatorname{Der}\mathbb{C}[t, t^{-1}] = \mathbb{C}\{t^{m+1}\frac{d}{dt} \mid m \in \mathbb{Z}\}$ (also denote by $\mathrm{Vect}(S^1)$, the Lie algebra of all vector fields on the circle).

$$\mathrm{Vir} = \operatorname{Der}\widehat{\mathbb{C}[t, t^{-1}]}.$$

The twisted Heisenberg–Virasoro algebra $\mathcal{H}$ was first studied by Arbarello et al. in [9], where a connection is established between the second cohomology of certain moduli spaces of curves and the second cohomology of the Lie algebra of differential operators of order at most one. By definition, $\mathcal{H}$ is the universal central extension of the following Lie algebra $\mathcal{D}$, which is the Lie algebra of differential operators order at most one.

Definition 2.1. As a vector space over $\mathbb{C}$, the Lie algebra $\mathcal{D}$ has a basis $\{d_n, Y_n \mid n \in \mathbb{Z}\}$ with the following relations

$$[d_m, d_n] = (n-m)d_{m+n}, \tag{2.2}$$

$$[d_m, Y_n] = nY_{m+n}, \tag{2.3}$$

$$[Y_m, Y_n] = 0, \tag{2.4}$$

for all $m,\ n \in \mathbb{Z}$.

Clearly, the centerless Heisenberg algebra $H = \mathbb{C}\{Y_m \mid m \in \mathbb{Z}\}$ and the Witt algebra (or centerless Virasoro algebra) $W = \mathbb{C}\{d_m \mid m \in \mathbb{Z}\}$ are subalgebras of $\mathcal{D}$.

2.2. Affine-Virasoro algebra

Definition 2.2. Let L be a finite dimensional Lie algebra with a nondegenerate invariant normalized symmetric bilinear form $(\ ,\ )$, then the affine-Virasoro Lie algebra is the vector space

$$L_{av} = L \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}C \oplus \bigoplus_{i\in\mathbb{Z}} \mathbb{C}d_i,$$

with Lie bracket:

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + m(x, y)\delta_{m+n,0}C,$$

$$[d_i, d_j] = (j-i)d_{i+j} + \frac{1}{12}(j^3 - j)\delta_{i+j,0}C,$$

$$[d_i, x \otimes t^m] = mx \otimes t^{m+i}, \qquad [C, L_{av}] = 0,$$

where $x,\ y \in L, m, n, i, j \in \mathbb{Z}$ (if L has no such form, we set $(x, y) = 0$ for all $x, y \in L$).

Remark. If $L = \mathbb{C}e$ is one dimensional, then L_{av} is just the twisted Heisenberg–Virasoro algebra (one center element).

Now we only consider specially L as the simple Lie algebra $\mathfrak{sl}_2 = \mathbb{C}\{e, f, h\}$. Then by Definition 2.2, the corresponding affine-Virasoro algebra $\mathcal{L} := L_{av} = \mathbb{C}\{e_i, f_i, h_i, d_i, C \mid i \in \mathbb{Z}\}$, with Lie bracket:

$$[e_i, f_j] = h_{i+j} + i\delta_{i+j,0}C,$$

$$[h_i, e_j] = 2e_{i+j}, \qquad [h_i, f_j] = -2f_{i+j},$$

$$[d_i, d_j] = (j-i)d_{i+j} + \frac{1}{12}(j^3 - j)\delta_{i+j,0}C,$$

$$[d_i, h_j] = jh_{i+j}, \qquad [h_i, h_j] = 2i\delta_{i+j,0}C,$$

$$[d_i, e_j] = je_{i+j}, \qquad [d_i, f_j] = jf_{i+j}, \qquad [C, \mathcal{L}] = 0,$$

where $i, j \in \mathbb{Z}$.

Remark. In fact, $\mathcal{L}$ is the even part of the $N = 3$ superconformal algebra [10].