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Network bipartivity and the transportation efficiency of European passenger airlines

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HIGHLIGHTS

- A new way for calculating the bipartivity of networks is introduced.
- The bipartivity of the European Low Cost Carriers and Traditional airlines shows significant differences.
- Alliances and airline mergers decrease the bipartivity of the corresponding networks.
- Bipartivity is strongly correlated with the transportation efficiency of the European airlines.

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ABSTRACT

The analysis of the structural organization of the interaction network of a complex system is central to understand its functioning. Here, we focus on the analysis of the bipartivity of graphs. We first introduce a mathematical approach to quantify bipartivity and show its implementation in general and random graphs. Then, we tackle the analysis of the transportation networks of European airlines from the point of view of their bipartivity and observe significant differences between traditional and low cost carriers. Bipartivity shows also that alliances and major mergers of traditional airlines provide a way to reduce bipartivity which, in its turn, is closely related to an increase of the transportation efficiency.

1. Introduction

A fundamental characteristic of complex systems is that, in general, they are networked. Thus, complex networks, which represent the skeleton of such complex systems, are ubiquitous in many real-world scenarios, ranging from the biomolecular – those representing gene transcription, protein interactions, and metabolic reactions – to the social and infrastructural organization of modern society [1]. Mathematically speaking, these networks are graphs with the nodes representing the entities of the system and the edges representing the "relations" among those entities. From a structuralist point view of nature it could be claimed that a large proportion of the properties of these complex systems is determined by the structure of these networks. The question about what do we mean by "the structure" of these networks is a tricky one. This situation is reminiscent of Edgar Allan Poe's response about what is the structure of a strange ship: "*What she is not, I can easily perceive—what she is I fear it is impossible to say*" [2]. Then, the pragmatic approach used in network theory and beyond is to consider structural invariants which characterize some portions of this structure which in global terms scape to our formal definitions. That is the reason why we have such a large amount of structural invariants, i.e., numbers that characterize some properties of the network independently of the labeling of nodes and edges [3]. Such invariants include the average path length, clustering coefficients, densities, assortativity coefficients, and many more (see [1,3] for non-exhaustive lists).

The concept of network bipartivity is one that has given rise to some structural invariants to characterize how much bipartivity a network has. Bipartivity has long been studied in graph theory as a black-and-white concept. That is, just by considering that a graph is or is not bipartite. However, in the noughties there were three papers that attempted to characterize how much bipartivity a non-bipartite graph has. The pioneering work of Holme et al. proposed the first of such measures in 2003 by using computational

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methods [4]. In 2005 Estrada and Rodríguez-Velázquez applied spectral graph theory to develop a mathematical characterization of graph bipartivity [5]. The third work, published by Pisanski and Randić, uses a characterization of network cyclicity to account for bipartivity [6]. In particular the bipartivity index developed by Estrada and Rodríguez-Velázquez has found a different number of applications to network problems ranging from the analysis of fullerene stability to the structure of food webs (see for instance [7–10]). Although this index is mathematically appealing – it is based on well defined matrix functions of the adjacency matrix of a graph – it has some drawbacks. The most important one is that it is bounded between 0.5 and 1.0, which means that it has a very narrow range of values for the analysis of networks. Also importantly, it is based on two different kinds of matrix functions – the exponential and the hyperbolic cosine – which make its calculation computationally complicated from the point of view of using numerical methods for their calculations.

Here we propose a new mathematical approach to quantify the bipartivity of a network by considering a single matrix function—the exponential. This new index is ranged between 0 and 1. We show a few mathematical properties of this index for general networks as well as for random graphs. We then embarked on the study of the airline transportation networks in Europe to see how the different degrees of bipartivity affect their global efficiency in terms of the number of passengers transported and the number of hours flown. We show that the new bipartivity index accounts very well for the main characteristics of these European airline networks and allow us to understand the main structural differences between traditional and low-cost carriers operating in Europe.

2. Graph theoretical preliminaries

Here we present some definitions, notations, and properties which will be used in this work (see [1,3]). A *graph* $G = (V, E)$ is defined by a set of n nodes (vertices) V and a set of m edges $E = \{(u, v)|u, v \in V\}$ between the nodes. An edge is said to be *incident* to a vertex u if there exists a node $v \neq u$ such that either $(u, v) \in E$ or $(v, u) \in E$. The graph is said to be *undirected* if the edges are formed by unordered pairs of vertices. A *walk* of length k in G is a set of nodes $i_1, i_2, \ldots, i_k, i_{k+1}$ such that for all $1 \leq l \leq k$, $(i_l, i_{l+1}) \in E$. A *closed walk* is a walk for which $i_1 = i_{k+1}$. A *path* is a walk with no repeated nodes. A graph is *connected* if there is a path connecting every pair of nodes. A graph with unweighted edges, no self-loops (edges from a node to itself), and no multiple edges is said to be *simple*. Throughout this work, we will always consider undirected, simple, and connected networks. In this setting the matrix $A = (a_{uv})$, called the *adjacency matrix* of the graph, has entries

$$a_{uv} = \begin{cases} 1 & \text{if } (u, v) \in E \\ 0 & \text{otherwise} \end{cases} \quad \forall u, v \in V,$$

and, in the particular case of an undirected network as the ones studied here, the adjacency matrix of the graph is symmetric, $a_{uv} = a_{vu}$, and thus its eigenvalues are real. In the following we label the eigenvalues of A in non-increasing order: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The degree of a node k_i is the number of edges incident to that node. Since A is a real-valued, symmetric matrix, we can decompose A into $A = Q\Lambda Q^T$ where Λ is a diagonal matrix containing the eigenvalues of A and $Q = [\mathbf{q}_1, \ldots, \mathbf{q}_n]$ is orthogonal, where $\mathbf{q}_i$ is an eigenvector associated with λ_i.

The network density is given by:

$$\delta = \frac{2m}{n(n-1)},$$

where m is the number of edges. Here we will call, as usual in network theory, *average path length* the average of the shortest path distance in the graph:

$$\bar{l} = \frac{2}{n(n-1)} \sum_{u \neq v} d(u, v),$$

where $d(u, v)$ is the shortest path distance between the nodes u and v.

An important quantity for the current work is one defined for studying communication processes in networks, which is called communicability function [11]. In particular, the self-communicability function, also known as the subgraph centrality [12] of the corresponding node, is defined as follows. Let u be a node of G, then

$$G_{uu} = \sum_{k=0}^{\infty} \frac{\left(A^k\right)_{uu}}{k!} = (\exp(A))_{uu} = \sum_{k=1}^{n} e^{\lambda_k} \mathbf{q}_k^2(u).$$

The sum of all subgraph centralities in a network is nowadays known as the Estrada index of a graph [13–16], which is defined as

$$\mathrm{EE}(G) = \sum_{u=1}^{n} (\exp(A))_{uu} = tr(\exp(A)).$$

By the properties of the matrix exponential and of the trace of a matrix we can easily see that

$$\mathrm{EE}(G) = tr(\sinh(A)) + tr(\cosh(A)).$$

These functions count the total number of closed walks starting (and ending) at node u, weighted in decreasing order of their length k by a factor $\frac{1}{k!}$; therefore it is considering shorter closed walks more influential than longer ones (see [11,17,18]). In particular, the hyperbolic sine function counts the number of closed walks of odd length in the graph and the hyperbolic cosine one counts the even-length ones.

3. Spectral bipartivity index in graphs

There are a few characterizations of bipartite graphs in graph theory. For instance, the following characterization is a well-known one [19].

Lemma 1. *A graph is bipartite if and only if it does not contain any odd cycle.*

We now provide a related characterization of bipartite graphs which will be of great usefulness in this work.

Theorem 2. *A graph is bipartite if and only if* $tr \sinh(A) = 0$.

Proof. Let us consider the Taylor series expansion of $tr \sinh(A)$

$$tr \sinh(A) = trA + \frac{trA^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{tr\left(A^{2k+1}\right)}{(2k+1)!}.$$

We know that $tr\left(A^{2k+1}\right)$ counts the number of closed walks of length $2k+1$ in the graph. Every closed walk of odd length necessarily involves an odd cycle. Then, because in a bipartite graph there are no odd cycles, $tr\left(A^{2k+1}\right) = 0$ for all k, which proves the above result. □

Let us call *frustrated* closed walk a closed walk which involves any odd cycle in the network. Similarly, a non-frustrated closed walk is the one which does not involve any odd cycle. Let us now consider a normalized measure of the difference between the number of non-frustrated and frustrated walks:

$$b_s = \frac{W^N - W^F}{W^N + W^F}. \quad (1)$$

The use of the term *frustrated* to designate closed walks involving any odd cycle in the network comes from its use in spin