Journal of Geometry and Physics 103 (2016) 37–42

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Semistability of certain bundles on second symmetric power of a curve

Krishanu Dan [a,*], Sarbeswar Pal [b]

[a] *Institute of Mathematical Sciences, C.I.T Campus, Tharamani, Chennai-600113, India*
[b] *Indian Statistical Institute, 8th Mile, Mysore Road, Bangalore-560059, India*

ARTICLE INFO

Article history:
Received 9 April 2015
Received in revised form 17 August 2015
Accepted 25 January 2016
Available online 4 February 2016

Keywords:
Vector bundles
Symmetric power
Semistability

ABSTRACT

Let C be a smooth irreducible projective curve and E be a stable bundle of rank 2 on C. Then one can associate a rank 4 vector bundle $\mathcal{F}_2(E)$ on $S^2(C)$, the second symmetric power of C. Our goal in this article is to study semistability of this bundle.

1. Introduction

It has been an interesting and important object to study vector bundles over smooth projective varieties. The moduli space of semistable vector bundles with fixed topological invariants is well understood for the case of curves. However the question of existence of such bundles is open for higher dimensional varieties. In this article we will study the semistability of certain vector bundles on second symmetric power of a smooth projective curve, which arises naturally.

Let C be smooth irreducible projective curve over the field $\mathbb{C}$ of complex numbers and E be a rank r vector bundle on C. There is a naturally associated vector bundle $\mathcal{F}_2(E)$ of rank $2r$ on the second symmetric power $S^2(C)$ which is defined in Section 2. The stability and semi-stability for case $r = 1$, i.e. when E is a line bundle on C, has been studied and well understood [1–3]. In this article we consider the case when rank E is two.

Fixing a point $x \in C$, the image of $\{x\} \times C$ in $S^2(C)$ defines an ample divisor H' on $S^2(C)$, which we denote by $x + C$. We prove the following:

Theorem 1.1. *Let E be a rank two stable vector bundle of even degree $d \geq 2$ on C such that $\mathcal{F}_2(E)$ is globally generated. Then the bundle $\mathcal{F}_2(E)$ on $S^2(C)$ is $\mu_{H'}$-semistable with respect to the ample class $H' = x + C$.*

Theorem 1.2. *Assume the genus of C greater than 2. Let E be a rank two $(0, 1)$-stable bundle (defined in Section 4) of odd degree $d \geq 1$ on C such that $\mathcal{F}_2(E)$ is globally generated. Then the bundle $\mathcal{F}_2(E)$ on $S^2(C)$ is $\mu_{H'}$-semistable with respect to the ample class $H' = x + C$.*

2. Preliminaries

Let C be a smooth irreducible projective curve over the field of complex numbers $\mathbb{C}$ of genus g. On the space $C \times C$, consider the following involution $C \times C \longrightarrow C \times C$, $(x, y) \mapsto (y, x)$. The resulting quotient space is denoted by $S^2(C)$, called

* Corresponding author.
E-mail addresses: bubukrish@gmail.com (K. Dan), sarbeswar11@gmail.com (S. Pal).

http://dx.doi.org/10.1016/j.geomphys.2016.01.009

the second symmetric power of C. It is a smooth irreducible projective surface over $\mathbb{C}$. Note that, $S^2(C)$ is naturally identified with the set of all degree 2 effective divisors of C. Set

$$\Delta_2 := \{(D, p) \in S^2(C) \times C | D = p + q, \text{ for some } q \in C\}.$$

Then Δ_2 is a divisor in $S^2(C) \times C$, called the universal divisor of degree 2. Let q_1 and q_2 be the projections from $S^2(C) \times C$ onto the first and second factors respectively. Then the restriction of the first projection to Δ_2 induces a morphism

$$q : \Delta_2 \longrightarrow S^2(C),$$

which is a two sheeted ramified covering. For any vector bundle E of rank r on C we construct a bundle $\mathcal{F}_2(E) := (q)_*(q_2^*(E)|_{\Delta_2})$ of rank $2r$ over $S^2(C)$. From the exact sequence

$$0 \to \mathcal{O}_{S^2(C)\times C}(-\Delta_2) \to \mathcal{O}_{S^2(C)\times C} \to \mathcal{O}_{\Delta_2} \to 0$$

on $S^2(C) \times C$ we get the following exact sequence on $S^2(C)$

$$0 \to q_{1*}(q_2^*E \otimes \mathcal{O}_{S^2(C)\times C}(-\Delta_2)) \to q_{1*}q_2^*E \to \mathcal{F}_2(E).$$

Define $f : C \times C \to \Delta_2$ by $(x, y) \mapsto (x + y, x)$. Then f gives an identification between $C \times C$ and Δ_2. Let $p_i : C \times C \to C$ be the ith coordinate projection and let $\pi : C \times C \to S^2(C)$ be the quotient map. Then it is easy to check that $\pi = q \circ f$ and $\mathcal{F}_2(E) = \pi_* p_2^* E$.

Remark 2.1. Let C be a smooth irreducible projective curve over $\mathbb{C}$ of genus g and let M be a line bundle on C of degree d. Consider the rank two vector bundle $\mathcal{F}_2(M) := \pi_* p_2^* M$ on $S^2(C)$. Using Grothendieck–Riemann–Roch, one can compute the Chern classes of $\mathcal{F}_2(M)$:

$$c_1(\mathcal{F}_2(M)) = (d - g - 1)x + \theta$$

and

$$c_2(\mathcal{F}_2(M)) = \binom{d-g}{2} x^2 + (d - g)x.\theta + \frac{\theta^2}{2}$$

where x is the image of the cohomology class of $x + C$ in $S^2(C)$, θ is the cohomology class of the pull back of the theta divisor in $\mathrm{Pic}^2(C)$ under the natural map of $S^2(C)$ to $\mathrm{Pic}^2(C)$ [4, Lemma 2.5, Chapter VIII]. Note that the cohomology group $H^4(S^2(C), \mathbb{Z})$ is naturally isomorphic to $\mathbb{Z}$, and $x^2 = 1$, $x.\theta = g$, $\theta^2 = g(g - 1)$.

To find the Chern character of $\mathcal{F}_2(E)$, for any rank r vector bundle E, first choose a filtration of E such that the successive quotients are line bundles and use the fact that $\mathcal{F}_2(\oplus M_k) = \oplus \mathcal{F}_2(M_k)$ where M_k's are line bundles over C. Then the Chern character of $\mathcal{F}_2(E)$ has the following expression [5]:

$$ch(\mathcal{F}_2(E)) = \mathrm{degree}(E)(1 - \exp(-x)) - r(g - 1) + r(1 + g + \theta)\exp(-x).$$

From the above expression one can easily see that $c_1(\mathcal{F}_2(E)) = (d - r(g + 1))x + r\theta$, where $d = \mathrm{degree}\, E$.

3. Semistability of $\mathcal{F}_2(E)$, for degree E even

Let C be a smooth irreducible projective curve over the field of complex numbers $\mathbb{C}$ of genus g and let E be a rank r vector bundle on C. In this section we will prove the semistability of the vector bundle $\mathcal{F}_2(E)$, when $r = 2$ and degree E is even. We start with the following definitions.

Definition 3.1. Let C be a non-singular irreducible curve. For a vector bundle F on C we define

$$\mu(F) := \frac{\mathrm{degree}(F)}{\mathrm{rank}(F)}.$$

A vector bundle F on C is said to be semistable (respectively, stable) if for every subbundle F' of F we have

$$\mu(F') \le \mu(F)(\text{respectively, } \mu(F') < \mu(F)).$$

Definition 3.2. Let X be a smooth irreducible surface and let H be an ample divisor on X. For a coherent torsion free sheaf F on X, we set

$$\mu_H(F) := \frac{\mathrm{degree}_H(F)}{\mathrm{rank}(F)}$$

where $\mathrm{degree}_H(F) = c_1(F) \cdot H$.

A vector bundle F on X is said to be μ_H-semistable (respectively, μ_H-stable), if for every coherent torsion free subsheaf F' of F with $0 < \mathrm{rank}(F') < \mathrm{rank}(F)$, we have

$$\mu_H(F') \le \mu_H(F)(\text{respectively, } \mu_H(F') < \mu_H(F)).$$