Chaos, Solitons and Fractals 89 (2016) 228–231

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Zero-Hopf bifurcation in the generalized Michelson system

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ARTICLE INFO

Article history:
Available online 28 November 2015

MSC:
37G15
37C80
37C30

Keywords:
Periodic solution
Averaging theory
Zero–Hopf Bifurcation
Michelson system
Triple-zero bifurcation

ABSTRACT

We provide sufficient conditions for the existence of two periodic solutions bifurcating from a zero–Hopf equilibrium for the differential system

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = a + by + cz - x^2/2,$$

where *a*, *b* and *c* are real arbitrary parameters. The regular perturbation of this differential system provides the normal form of the so–called triple–zero bifurcation.

1. Introduction and statement of the main results

In the paper [1] it is proved that the normal form of the triple–zero bifurcation can be understood as a regular perturbation of the following generalized Michelson system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= a + by + cz - x^2/2, \end{aligned} \qquad (1)$$

where *a*, *b* and *c* are real arbitrary parameters. The prime denotes derivative with respect to the independent variable *t*.

A *zero–Hopf equilibrium* is an equilibrium point of a 3–dimensional autonomous differential system, which has a zero eigenvalue and a pair of purely imaginary eigenvalues. Usually the *zero–Hopf bifurcation* is a two–parameter unfolding of a 3-dimensional autonomous differential system with a zero–Hopf equilibrium. The unfolding has an isolated equilibrium with a zero eigenvalue and a pair of purely imaginary eigenvalues if the two parameters take zero values, and the unfolding has different topological type of dynamics in the small neighborhood of this isolated equilibrium as the two parameters vary in a small neighborhood of the origin. For instance this zero–Hopf bifurcation has been studied in [6–9,11], and it has been shown that some complicated invariant sets of the unfolding could be bifurcated from the isolated zero–Hopf equilibrium under some conditions. Hence, in some cases zero–Hopf bifurcation could imply a local birth of "chaos" see for instance the articles of [2–5,11]).

Our objective is to study analytically the periodic solutions of the zero–Hopf bifurcation for the generalized Michelson differential system (1). In the previous mentioned papers on the zero-Hopf bifurcation they do not use averaging theory for studying such kind of bifurcation. Our goal is to study analytically such a bifurcation using averaging theory which will allows to provide an explicit expression of the dominant terms of the periodic solution bifurcating from the zero-Hopf equilibrium. First, in the next proposition we characterize when the equilibrium point of the generalized Michelson system (1) is a zero-Hopf equilibrium point.

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http://dx.doi.org/10.1016/j.chaos.2015.11.013

Proposition 1. *There is an one–parameter family of the generalized Michelson system (1) for which the origin of coordinates is a zero–Hopf equilibrium point. Namely* $a = 0$, $b = -\omega^2$, $c = 0$.

Proposition 1 is proved in Section 2.

Theorem 2. *Assume that in the generalized Michelson system (1) we have*

$$a = \varepsilon^2(a_2 + \omega^4/2), \quad a_2 > 0, \quad b = -\omega^2 + \varepsilon b_1, \quad \omega > 0,$$
$$c = \varepsilon. \tag{2}$$

Then for $\varepsilon \neq 0$ *sufficiently small system (1) has two periodic solutions* $(x_i(t, \varepsilon), y_i(t, \varepsilon), z_i(t, \varepsilon))$ *bifurcating from the zero–Hopf equilibrium of Proposition 1, namely*

$$\left(\varepsilon \frac{V_i^* - \omega R^* \cos(\omega t) - \omega^2\sqrt{2a_2 + \omega^4}}{\omega^2} + O(\varepsilon^2),\right.$$
$$\left. \varepsilon R^* \sin(\omega t) + O(\varepsilon^2), \quad \varepsilon\omega R^* \cos(\omega t) + O(\varepsilon^2)\right), \tag{3}$$

where

$$R^* = 2\sqrt{a_2}\omega,$$
$$V_i^* = \omega^2((-1)^i\sqrt{2a_2 + \omega^4} - \omega^2) \quad \text{for } i = 1, 2.$$

Moreover, these two periodic solutions are unstable.

Theorem 2 improves and extends the result of [10] where only one periodic solution was detected bifurcating from the zero–Hopf equilibrium for a subsystem of system (1).

2. Proof of Proposition 1 and Theorem 2

Proof of Proposition 1. System (1) possesses the equilibrium points $(x, y, z) = (\pm\sqrt{2a}, 0, 0)$ if $a \geq 0$. The Jacobian matrix of system (1) at the equilibrium point $(\pm\sqrt{2a}, 0, 0)$ is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \pm\sqrt{2a} & b & c \end{pmatrix}.$$

Its characteristic polynomial is $p(\lambda) = -\lambda^3 + c\lambda^2 + b\lambda \pm \sqrt{2a}$. In order to study the zero-Hopf bifurcation we force that $p(\lambda) = -(\lambda - \varepsilon)(\lambda^2 + \omega^2)$. $p(\lambda) + (\lambda - \varepsilon)(\lambda^2 + \omega^2) = 0$. This occurs if and only if the coefficients of this equation are $\pm\sqrt{2a} - \varepsilon\omega^2 = 0$, $b + \omega^2 = 0$, $c - \varepsilon = 0$. We obtain $a = \varepsilon^2 \frac{\omega^4}{2}$, $b = -\omega^2$, $c = \varepsilon$. This completes the proof of the proposition. □

The differential system (1) satisfying (2) has two equilibria, namely

$$p_\pm = (\pm\varepsilon\sqrt{2a_2 + \omega^4}, 0, 0).$$

First we study the periodic solutions bifurcating from the zero–Hopf equilibrium near the equilibrium p_-.

For applying the averaging theory described in the Appendix to system (1) satisfying (2) we translate the equilibrium point p_- to the origin by doing the change of variables

$$(x, y, z) = (x_1 - \varepsilon\sqrt{2a_2 + \omega^4}, y_1, z_1). \tag{4}$$

The differential system in the new variables (x_1, y_1, z_1) is

$$\dot{x}_1 = y_1,$$
$$\dot{y}_1 = z_1,$$
$$\dot{z}_1 = -\omega^2 y_1 - \frac{x_1^2}{2} + \varepsilon\left(b_1 y_1 + z_1 + x_1\sqrt{2a_2 + \omega^4}\right). \tag{5}$$

We need to write the linear part of system (5) at the equilibrium point $(0, 0, 0)$ in its real Jordan normal form, i.e. into the form

$$\begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

in order to facilitate the application of the averaging theory, given by Theorem 3, for computing the zero–Hopf bifurcation. Then, doing the change of variables $(x_1, y_1, z_1) \to (X, Y, Z)$ given by

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{\omega} \\ 0 & 1 & 0 \\ \omega^2 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix},$$

the differential system (5) having its linear part in its real Jordan form is

$$\dot{X} = -\frac{2Y\omega^6 + (Z - \omega X)^2}{2\omega^5} + \varepsilon\left(X + \frac{b_1 Y}{\omega} + \frac{1}{\omega^3}(Z - X\omega)\sqrt{2a_2 + \omega^4}\right),$$
$$\dot{Y} = \omega X,$$
$$\dot{Z} = -\frac{(Z - \omega X)^2}{2\omega^4} + \varepsilon\left(\omega X + b_1 Y + \frac{1}{\omega^2}(Z - \omega X)\sqrt{2a_2 + \omega^4}\right). \tag{6}$$

Consider the cylindrical coordinates (r, θ, Z) defined by $X = r\cos\theta$, $Y = r\sin\theta$, $Z = Z$ then the differential system (6) becomes

$$\dot{r} = -\frac{\cos\theta(Z - \omega r\cos\theta)^2}{2\omega^5} + \varepsilon\left[\frac{r\cos\theta(\omega\cos\theta + b_1\sin\theta)}{\omega} - \frac{1}{\omega^3}\cos\theta(Z - \omega r\cos\theta)\sqrt{2a_2 + \omega^4}\right],$$
$$\dot{\theta} = \frac{2r\omega^6 + (Z - r\omega\cos\theta)^2\sin\theta}{2r\omega^5} - \varepsilon\left[\frac{\sin\theta(\omega\cos\theta + b_1\sin\theta)}{\omega} - \frac{\sqrt{2a_2 + \omega^4}}{\omega^3}\left[\frac{Z\sin\theta}{r} - \frac{\omega\sin(2\theta)}{2}\right]\right],$$
$$\dot{Z} = -\frac{(Z - r\omega\cos\theta)^2}{2\omega^4} + \varepsilon\left[\omega r\cos\theta + b_1 r\sin\theta + \frac{1}{\omega^2}(Z - \omega r\cos\theta)\sqrt{2a_2 + \omega^4}\right]. \tag{7}$$