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Prioritary omalous bundles on Hirzebruch surfaces

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To Professor Vasile Brinzanescu on the occasion of his 70th birthday, with admiration

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ABSTRACT

An irreducible algebraic stack is called *unirational* if there exists a surjective morphism, representable by algebraic spaces, from a rational variety to an open substack. We prove unirationality of the stack of prioritary omalous bundles on Hirzebruch surfaces, which implies also the unirationality of the moduli space of omalous H-stable bundles for any ample line bundle H on a Hirzebruch surface (compare with Costa and Miro-Ŕoig, 2002). To this end, we find an explicit description of the duals of omalous rank-two bundles with a vanishing condition in terms of monads. Since these bundles are prioritary, we conclude that the stack of prioritary omalous bundles on a Hirzebruch surface different from $\mathbb{P}^1 \times \mathbb{P}^1$ is dominated by an irreducible section of a Segre variety, and this linear section is rational (Ionescu, 2015). In the case of the space quadric, the stack has been explicitly described by N. Buchdahl. As a main tool we use Buchdahl's Beilinson-type spectral sequence. Monad descriptions of omalous bundles on hypersurfaces in $\mathbb{P}^4$, Calabi–Yau complete intersection, blowups of the projective plane and Segre varieties have been recently obtained by A.A. Henni and M. Jardim (Henni and Jardim, 2013), and monads on Hirzebruch surfaces have been applied in a different context in Bartocci et al. (2015).

1. Introduction

An *omalous bundle* on a complex projective smooth variety is a vector bundle whose determinant is the anti-canonical bundle and with second Chern class equal to the class of the tangent bundle. Omalous bundles have been introduced by Ron Donagi and the motivation comes from physics: these conditions on the Chern classes imply the usual Green–Schwartz anomaly cancellation conditions. The omality condition is used in the construction of quantum sheaf cohomology [1]. In [2] A.A. Henni and M. Jardim found explicit descriptions of (stable) omalous bundles on several types of varieties: hypersurfaces in $\mathbb{P}^4$, Calabi–Yau complete intersection, blowups of the projective plane and Segre varieties.

Prioritary sheaves on the projective plane were defined by A. Hirschowitz and Y. Laszlo in [3]. This notion was extended on birationally ruled surfaces by Ch. Walter in [4,5]. In [4] it is proved that if H is a polarisation on a birationally ruled surface with a numerical condition (such polarisations always exist), then any H-semistable torsion-free sheaf is prioritary, hence this notion extends H-semistability. The advantage in working with prioritary sheaves instead of (semi) stable ones is that their definition is polarisation-free. The most important result on prioritary sheaves is the irreducibility and the smoothness of the stack [3–5].

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The aim of this paper is to prove unirationality of the stack of prioritary omalous bundles on Hirzebruch surfaces via an explicit monad description of prioritary omalous rank-two vector bundles with a vanishing condition on a Hirzebruch surface. For technical reasons, we work with duals of omalous bundles. This approach does not affect the result, as the dual of a prioritary rank-two bundle is also prioritary.

We work over the field of complex numbers. The outline of the paper is the following. In Section 2, we set the notation and we recall some facts that will be used in the core of the paper. We discuss the numerical invariants associated to rank-two vector bundles on Hirzebruch surfaces and the canonical extensions [6,7], Beilinson spectral sequences on Hirzebruch surfaces [8], and the general theory of monads [9]. In Section 3 we describe completely the duals of prioritary omalous bundles in terms of the associated numerical invariants, Proposition 1. A similar description is valid for bundles with arbitrary Chern classes, Remark 1. In Section 4 we find a necessary and sufficient condition, given by the vanishing of the space of sections of a suitable twist for a dual of an omalous bundle to be given by a specific monad, Theorem 2. This vanishing condition is satisfied by all the duals of (semi) stable omalous bundles and moreover the duals of omalous bundles that satisfy these condition are necessary prioritary, Proposition 3. Hence the stack of bundles with this condition sits between all the stacks of (semi) stable bundles and the stack of prioritary bundles, and the inclusions are strict, Remark 3. The monad description obtained in Theorem 2 is used in Section 5 to prove that the stack of prioritary omalous bundles is dominated by a rational variety, which is a linear section of a Segre variety, Theorem 3. Hence the stack of prioritary omalous bundles is an irreducible, smooth, unirational stack of dimension 4, Theorem 3. Consequently, all the moduli spaces of stable omalous bundles are unirational, Corollary 1, see also [10].

2. Preliminaries

In this preliminary section we fix the notation and we recall some definitions and facts that will be used in the main Sections 3 and 4.

2.1. Hirzebruch surfaces

Let $X = \Sigma_e$ be a Hirzebruch surface, $\Sigma_e = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)) \xrightarrow{\pi} \mathbb{P}^1$ with $e \geq 0$. Denote by $C_0 = \mathcal{O}_X(1)$ the negative section ($C_0^2 = -e$), and by F a fibre of the ruling ($C_0 \cdot F = 1, F^2 = 0$). Recall that $\mathrm{Pic}(X) = \mathbb{Z} \cdot C_0 \oplus \mathbb{Z} \cdot F$ and the canonical bundle is $K_X = \mathcal{O}_X(-2C_0 - (e+2)F)$. A line bundle $\mathcal{O}_X(aC_0 + bF)$ has a nonzero global section if and only if $a \geq 0$ and $b \geq 0$. In what concerns the cohomology groups H^1 they are given by

Lemma 1. *Let $X = \Sigma_e$ be a Hirzebruch surface and $a, b \in \mathbb{Z}$. Then*

$$H^1(X, \mathcal{O}_X(aC_0 + bF)) \cong \begin{cases} H^0\left(\mathbb{P}^1, \bigoplus_{k=1}^{-a-1} \mathcal{O}_{\mathbb{P}^1}(ke+b)\right), & \text{if } a \leq -2 \\ 0, & \text{if } a = -1 \\ H^0\left(\mathbb{P}^1, \bigoplus_{k=0}^{a} \mathcal{O}_{\mathbb{P}^1}(ke-b-2)\right), & \text{if } a \geq 0. \end{cases}$$

2.2. Rank-two vector bundles on Hirzebruch surfaces as extensions

In this section we recall from [6] and [7] the numerical invariants naturally associated to rank-two bundles on Hirzebruch surfaces and the canonical extensions.

Let V be a rank-two vector bundle on a Hirzebruch surface $X = \Sigma_e \xrightarrow{\pi} \mathbb{P}^1$ with Chern classes $c_1(V) = \alpha C_0 + \beta F$ and $c_2(V) = c_2 \in \mathbb{Z}$. Since the fibres of the ruling are projective lines, we can speak about the generic splitting type of V i.e.:

$$V|_F \cong \mathcal{O}_F(d) \oplus \mathcal{O}_F(\alpha - d)$$

for a general fibre F, where $2d \geq \alpha$. The integer d is the first numerical invariant of V.

The second numerical invariant r is obtained from a normalisation process:

$$r = \max\{\ell \in \mathbb{Z} \mid H^0(X, V(-dC_0 - \ell F)) \neq 0\}.$$

In this context, we have the following result [6,7] (see also [11] Chapter 6):

Theorem 1. *Notation as above. There exists ζ a zero-dimensional locally complete intersection subscheme of X (or the empty set) of length $\ell(\zeta) := c_2 + \alpha(de - r) - \beta d + 2dr - d^2 e \geq 0$ such that V is presented as an extension:*

$$0 \to \mathcal{O}_X(dC_0 + rF) \to V \to \mathcal{O}_X((\alpha - d)C_0 + (\beta - r)F) \otimes \mathcal{I}_\zeta \to 0. \tag{1}$$

The extension (1) is called the *canonical extension* of V. This extension and the invariants d and r are very useful in a number of situations. For example, in [12] a numerical stability criterion involving these invariants has been proved. The existence of vector bundles with given numerical invariants has been settled in [13].