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Gauging without initial symmetry

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ABSTRACT

The gauge principle is at the heart of a good part of fundamental physics: Starting with a group G of so-called rigid symmetries of a functional defined over space–time Σ, the original functional is extended appropriately by additional Lie(G)-valued 1-form gauge fields so as to lift the symmetry to Maps(Σ, G). Physically relevant quantities are then to be obtained as the quotient of the solutions to the Euler–Lagrange equations by these gauge symmetries.

In this article we show that one can construct a gauge theory for a standard sigma model in arbitrary space–time dimensions where the target metric is not invariant with respect to *any* rigid symmetry group, but satisfies a much weaker condition: It is sufficient to find a collection of vector fields v_a on the target M satisfying the *extended Killing equation* $v_{a(i;j)} = 0$ for *some* connection acting on the index a. For regular foliations this is equivalent to requiring the conormal bundle to the leaves with its induced metric to be invariant under leaf-preserving diffeomorphisms of M, which in turn generalizes Riemannian submersions to which the notion reduces for smooth leaf spaces $M/\sim$.

The resulting gauge theory has the usual quotient effect with respect to the original ungauged theory: in this way, much more general orbits can be factored out than usually considered. In some cases these are orbits that do not correspond to an initial symmetry, but still can be generated by a finite-dimensional Lie group G. Then the presented gauging procedure leads to an ordinary gauge theory with Lie algebra valued 1-form gauge fields, but showing an unconventional transformation law. In general, however, one finds that the notion of an ordinary structural Lie group is too restrictive and should be replaced by the much more general notion of a structural Lie groupoid.

1. Introduction

The Standard Model of elementary particle physics, but also General Relativity and String Theory, are gauge theories. In the former case, for example, gauging of an SU(3) rigid symmetry rotation between the three quarks leads to the introduction of the eight gluons that mediate the interaction between those elementary particles. Mathematically the resulting theory is described by connections in a principle bundle (in the above example with the structure group SU(3), the connection 1-forms representing the dim SU(3) $=$ 8 gluons) with the matter fields being sections in appropriate associated vector bundles.

The procedure can be generalized to matter fields being sections in arbitrary associated fiber bundles, the fibers being equipped with geometric structures invariant w.r.t. some group G, the structural or “rigid” symmetry group. In the case of

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a trivial bundle, sections correspond to maps from the base manifold Σ to the fiber M and one obtains a sigma model, such as e.g. the "standard" one:

$$S_0[X] = \int_\Sigma \frac{1}{2} g_{ij}(X)\, \mathrm{d}X^i \wedge *\mathrm{d}X^j. \tag{1}$$

This is a functional on smooth maps $X \colon \Sigma \to M$. The d-dimensional spacetime Σ and the n-dimensional target manifold M carry a (possibly Lorentzian signature) metric γ and g, respectively, γ entering by means of $*$. The functional (1) is written as a functional of n scalar fields $X^i = X^*(x^i)$, corresponding to a choice of local coordinates $(x^i)_{i=1}^n$ on M.

Symmetries of the geometrical data on the source manifold Σ or on the target manifold M lift to symmetries of functionals using only such data. So, in the case of (1) an invariance of h and g leads to an invariance of S_0. By the Noether procedure this gives rise to conserved quantities. For example, if h is a flat, these conserved quantities yield the energy momentum tensor $T^{\mu\nu}$.

Let us now suppose that the metric g has a nontrivial isometry group G, which infinitesimally implies $\mathcal{L}_v g = 0$, valid for the vector fields $v = \rho(\xi)$ on M corresponding to arbitrary elements $\xi \in \mathfrak{g} = \mathrm{Lie}(G)$. In this case, there is a canonical procedure to lift the induced G-symmetry of S_0 to a gauge symmetry on an extended functional S_1 called *minimal coupling*: After introducing $\mathfrak{g}$-valued 1-forms $A = A^a \xi_a \in \Omega^a(\Sigma, \mathfrak{g})$, ξ_a denoting any basis of $\mathfrak{g}$, ordinary derivatives $\mathrm{d}X^i$ on the scalar fields X^i are replaced by covariant ones,

$$\mathrm{D}X^i := \mathrm{d}X^i - \rho_a^i(X) A^a, \tag{2}$$

where $\rho_a^i(x)\partial_i \equiv \rho(\xi_a)$. The new functional

$$S[X, A] = \int_\Sigma \frac{1}{2} g_{ij}(X)\, \mathrm{D}X^i \wedge *\mathrm{D}X^j \tag{3}$$

is now invariant with respect to the combined infinitesimal gauge symmetries generated by

$$\delta X^i = \rho_a^i(X)\varepsilon^a, \tag{4}$$

$$\delta A^a = \mathrm{d}\varepsilon^a + C_{bc}^a A^b \varepsilon^c, \tag{5}$$

for arbitrary $\varepsilon^a \in C^\infty(\Sigma)$. Here C_{bc}^a are the structure constants of the Lie algebra $\mathfrak{g}$ in the chosen basis.

In the space of (pseudo) Riemannian metrics, those permitting a non-trivial invariance or isometry group G are the big exception. A generic metric g does not permit *any* non-vanishing vector field v satisfying $\mathcal{L}_v g = 0$.

It is conventional belief that an isometry is necessary to gauge the functional (1). It is our intention to show that this is far from true: First, there may be group actions on the target M that are *not isometries* but still can be gauged using Lie algebra valued 1-forms. Second, and maybe more important, one does not need to restrict to the action of finite-dimensional Lie groups. It is sufficient to have a *Lie groupoid* $\mathcal{G}$ over M. In fact, the use of Lie groupoids (and their associated Lie algebroids) in the context of gauge theories is even *suggested* by the present analysis as the much more generic one.

2. The case of 1-dimensional leaves

For a conceptual orientation, we first consider the highly simplified situation of a (regular) foliation of M into *one-dimensional*, hyper-surface-orthogonal leaves for a positive-definite metric g. In this case we can choose an adapted local coordinate system such that ∂_1 generates these leaves and $x^1 = const$ yields orthogonal hyper-planes. This implies that $g_{1i} = 0$ for all $i \neq 1$ or, if we denote those indices by Greek letters from the beginning of the alphabet, that $g_{1\alpha} = 0$ (while certainly $g_{11} > 0$). In this section we are going to explicitly assume that ∂_1 is *not* generating an isometry of g. This is tantamount to $g_{ij,1} \neq 0$, for at least some components of the matrix (g_{ij}).

According to standard folklore, it should not be possible to extend S_0 by gauge fields such that ∂_1 becomes a direction of gauge symmetries, i.e. such that $\delta X^1 = \varepsilon$ will leave the extended action invariant for an arbitrary choice of the parameter function $\varepsilon \in C^\infty(\Sigma)$ (together with an appropriate transformation of the gauge field).

Since the leaves are 1-dimensional, we will introduce also only one gauge field $A \in \Omega^1(\Sigma)$ and consider the action functional

$$S[X, A] = \int_\Sigma \frac{1}{2} g_{11}(X)\, (\mathrm{d}X^1 - A) \wedge *(\mathrm{d}X^1 - A) + \frac{1}{2} g_{\alpha\beta}(X)\, \mathrm{d}X^\alpha \wedge *\mathrm{d}X^\beta. \tag{6}$$

If we postulate the conventional $\delta A = \mathrm{d}\varepsilon$, we achieve that $\mathrm{d}X^1 - A$ is strictly gauge invariant. It is then easy to see that with this transformation of the gauge field, we necessarily need $g_{ij,1} = 0$, which would imply that ∂_1 generates isometries of g. However, one notices that changes of g_{11} under the flow of ∂_1 can be compensated by means of a modified transformation of the gauge field A. It is sufficient to require merely $g_{\alpha\beta,1} = 0$ for gauge invariance of (6) if $\delta X^1 = \varepsilon$ is amended by

$$\delta A = \mathrm{d}\varepsilon + \frac{\varepsilon}{2} \left(\ln(g_{11})\right)_{,1} (\mathrm{d}X^1 - A). \tag{7}$$