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# Towards a noncommutative Brouwer fixed-point theorem

Ludwik Dąbrowski

*SISSA (Scuola Internazionale Superiore di Studi Avanzati), Via Bonomea 265, 34136 Trieste, Italy*

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ABSTRACT

We present some results and conjectures on a generalization to the noncommutative setup of the Brouwer fixed-point theorem from the Borsuk–Ulam theorem perspective.



## 1. Introduction

The Borsuk–Ulam theorem [1], a fundamental theorem of topology, is often formulated in one of three equivalent versions, that regard continuous maps whose either domain, or codomain, are spheres.

**Theorem 1.1.** *For any $n \in \mathbb{N}$ let $\sigma : x \mapsto -x$ be the antipodal involution of $S^n$. Then:*

(0) *If $F: S^n \to \mathbb{R}^n$ is continuous, then there exists $x \in S^n$, such that $F(x) = F(\sigma(x))$.*
(i) *There is no continuous map $f: S^n \to S^{n-1}$, such that $f \circ \sigma = \sigma \circ f$.*
(ii) *There is no continuous map $g : B^n \to S^{n-1}$, such that $g \circ \sigma(x) = \sigma \circ g(x)$ for all $x \in \partial B^n = S^{n-1}$.*
*The statements* (0), (i) *and* (ii) *are equivalent.* ◇

Here $S^n$ is the unit sphere in $\mathbb{R}^{n+1}$ and $B^n$ is the unit ball in $\mathbb{R}^n$, but homeomorphic spaces work as well. The antipodal involution $\sigma : x \mapsto -x$ generates a free action of the group $\mathbb{Z}_2$ and we call $\mathbb{Z}_2$-*equivariant* those maps that commute with $\sigma$.

The Borsuk–Ulam theorem has a large variety of proofs, nowadays usually employing the *degree* of a map (the case $n = 1$ is easily seen by the intermediate value theorem). Here we explain the equivalence of (0) and (i) in Theorem 1.1. Indeed, the logical negation of (0) would provide a map given by

$$f(x) := \frac{F(x) - F(-x)}{\|F(x) - F(-x)\|}, \tag{1.1}$$

contradicting (i). Conversely, $f$ viewed as a map into $\mathbb{R}^n$ would provide a counterexample to Theorem 1.1(0). Instead the proof of the equivalence of (i) and (ii) will be a special instance of the proof we shall give of the more general Proposition 2.1.

The Borsuk–Ulam theorem has lot of applications to differential equations, combinatorics (e.g. partitioning, necklace division), Nash equilibria, and others, see e.g. [2].

The well known equivalent theorems are the Lusternik-Schnirelmann theorem (that at least one among $n + 1$ open or closed sets covering $S^n$ contains a pair of antipodal points), and combinatorial Tucker's lemma and Fan's lemma [3].

*E-mail address:* dabrow@sissa.it.

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There is also plentiful of corollaries. Some of them are "fun facts", e.g. the case $n = 2$ is often illustrated by saying that at any moment there is always a pair of antipodal points on the Earth's surface with equal temperature and pressure. But also, that squashing (with folding admitted) a balloon onto the floor there always is a pair of antipodal points one on the top of the other.

A famous corollary known as *Ham Sandwich theorem* (sometimes named *Yolk, white and the shell of an egg* theorem) states that for any compact sets $V_1, \dots, V_n$ in $\mathbb{R}^n$ we can always find a hyperplane dividing each of them in two subsets of equal volume. Another impressive implication is that no subset of $\mathbb{R}^n$ is homeomorphic to $S^n$. Also the famous Brouwer fixed-point theorem, which can be formulated in two equivalent versions, is a corollary.

**Theorem 1.2.** (I) *A continuous map from the ball $B^n$ to itself has a fixed point.*
(II) *There is no continuous map $g : B^n \to S^{n-1}$ that is the identity on the boundary $\partial B^n = S^{n-1}$.*

*The statements* (I) *and* (II) *are equivalent and are corollary of* Theorem 1.1(ii). ◇

**Proof.** We show the equivalence of (I) and (II), by showing the equivalence of their logical negations.

If there is $g : B^n \to S^{n-1}$ such that $g|_{S^{n-1}} = \mathrm{id}$, then, $\sigma \circ g$ would have no fixed point, which shows (I)⇒(II).

Next, assume there exists $h : B^n \to B^n$ such that $h(x) \neq x$, $\forall x \in B^n$. Then $g(x)$, defined as the intersection point of the half-line passing from $h(x)$ through $x$ with $\partial B^n = S^{n-1}$, would define a continuous map $g : B^n \to S^{n-1}$ such that $g|_{S^{n-1}} = \mathrm{id}$, which shows (II)⇒(I).

Finally note that $g$ as above is in particular $\mathbb{Z}_2$ equivariant on $S^{n-1}$, which shows indirectly that (II) (and thus also (I)) is a corollary of Theorem 1.1(ii). □

This theorem can in turn be exemplified for $n = 2$ by recognizing that, say on a map of Lazio, placed on the table of the lecture hall at Villa Mondragone, there must be some point lying directly over the point that it represents. Instead the case $n = 3$ is usually illustrated by stirring a cup of coffee.

The Brouwer fixed-point theorem has lot of important applications too, and is also known to be equivalent to Knaster–Kuratowski–Mazurkiewicz lemma (for coverings), or to combinatorial Sperner's lemma (with a Fair Division result following).

## 2. Generalizations

There have been numerous generalizations and strengthenings of the Borsuk–Ulam theorem, see e.g. the comprehensive survey [4], with almost 500 references before 1985. For some more recent generalizations regarding the dimension of the coincidence set of the maps $f$ or $g$ for more general manifolds, or for homology spheres, and equivariance under other groups see e.g. [5] and [6], and references therein. Here we shall briefly present few generalizations, whose noncommutative analogues could be most accessible, in our opinion.

### *2.1. Going beyond spheres*

The version (0) of the Borsuk–Ulam Theorem 1.1 employs a linear structure, and is not clear how it generalizes to spaces more general than spheres. Instead the versions (i) and (ii) can indeed be generalized as follows.

For that view $S^n$ as the non reduced suspension $\Sigma S^{n-1}$ of $S^{n-1}$, i.e. the quotient of $[0, 1] \times S^{n-1}$ by the equivalence relation $R_\Sigma$ generated by

$$(0, x) \sim (0, x'), \qquad (1, x) \sim (1, x'). \tag{2.2}$$

In this homeomorphic realization the $\mathbb{Z}_2$-action becomes

$$(t, x) \mapsto (1 - t, -x). \tag{2.3}$$

Furthermore notice that the ball $B^n$ is homeomorphic to the cone $\Gamma S^{n-1}$ of $S^{n-1}$, i.e. the quotient of $[0, \frac{1}{2}] \times S^{n-1}$ by the equivalence relation $R_\Gamma$ generated by

$$(0, x) \sim (0, x'). \tag{2.4}$$

**Proposition 2.1.** *Let $X$ be a compact space $X$ of finite covering dimension with a free $\mathbb{Z}_2$-action given by an involution $\sigma$.*

(i) *There is no $\mathbb{Z}_2$-equivariant continuous map $f : \Sigma X \to X$, where the $\mathbb{Z}_2$-action on $\Sigma X$ is given by the involution*

$$(t, x) \mapsto (1 - t, \sigma(x)), \tag{2.5}$$

(ii) *There is no continuous map $g : \Gamma X \to X$ that is $\mathbb{Z}_2$-equivariant on $\partial(\Gamma X) = X$.*

*The conditions* (i) *and* (ii) *are equivalent.* ◇