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A statically compatible layerwise stress model for the analysis of multilayered plates

Rawad Baroud[a,b], Karam Sab[a,*], Jean-Franois Caron[a], Fouad Kaddah[b]

[a] Université Paris-Est, Laboratoire Navier, ENPC, IFSTTAR, CNRS UMR 8205, 6, 8 avenue Blaise Pascal, Marne-la-Vallée Cedex 2 77455, France
[b] Université Saint Joseph, École Suprieure d'Ingnieurs Beyrouth, Centre Libanais d'Études et de Recherches de la Construction, Mar Roukoz-Dekwaneh, Lebanon

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ABSTRACT

This paper proposes a new layerwise model for multilayered plates. The model, called *SCLS1* for Statically Compatible Layerwise Stresses with first-order membrane stress approximations per layer in thickness direction, complies exactly with the 3D equilibrium equations and with the free-edge boundary conditions. As in the *LS1* model initially proposed by Naciri et al. (1998), the laminated plate is considered as a superposition of Reissner plates coupled by interfacial stresses which are considered as generalized stresses. However, the divergences of the interlaminar transverse shears are introduced as additional generalized stresses in the *SCLS1*model. Also, a *refined* version of the new model is obtained by introducing several *mathematical layers* per physical layer. Contrary to the *LS1* model which is derived by means of the Hellinger–Reissner principle, the new model is derived by means of the minimum of the complementary energy principle. This ensures the convergence of the *refined SCLS1* solution to the exact 3D solution as the number of mathematical layers per physical layer increases. The new model has been implemented in a new version of the in-house finite element code *MPFEAP*. Several comparisons are made between *LS1, SCLS1* and full 3D FE models in order to assess the performances of the new model which shows itself very effective.

1. Introduction

Multilayered plates are commonly used in weight-sensitive industrial applications, such as aerospace and automotive structures and their excellent thermal characteristics and specific strengths make them very attractive even for civil applications. An efficient design of these structures requires dedicated numerical tools that cope with their peculiar heterogeneous structure and anisotropy. For the structural engineer, the multilayered plate is conveniently represented as a stack of homogeneous, anisotropic plies. One of the major issues in design and analysis of such plate is related to free-edge effects. It has been demonstrated that differences in elastic properties of adjacent layers generally result in a highly concentrated interlaminar stresses near free edges (Chue and Liu, 2002; Leguillon, 1999; Mittelstedta and Becker, 2005; Ting and Chou, 1981; Wang and Choi, 1982). This phenomenon can lead to interlaminar failures (delaminations) which may cause global failure of the multilayered structure.

Highly detailed three-dimensional (3D) models are usually very expensive in terms of computational time and memory. They are consequently confined to specific regions or for providing reference results for some specific configurations. By taking into account the relatively small thickness of multilayered structures, many reduced models that are capable to provide sufficient information within a computationally tractable two-dimensional (2D) model have been proposed in the literature for the analysis of multilayered structures. These 2D models can be classified as equivalent single layer (ESL) theories or layerwise theories.

In ESL theories, the multilayer is considered as a one-layer homogeneous plate with an equivalent global behavior. Therefore, the number of governing equations is independent of the number of plate layers. Classical laminate theory (CLT) based on Kirchhoff–Love hypotheses and first-order shear deformation theory (FSDT) based on Reissner–Mindlin assumptions are the most widely used ESL theories. Many other ESL models based on higher order theories have been proposed in the literature (Cecchi and Sab, 2007; Cho and Parmerter, 1993; Nguyen et al., 2008; Reddy, 1984; Swaminathan and Ragounadin, 2004; Whitney and Sun, 1973). Two main paths were followed for deriving ESL models suitable for laminated plates: axiomatic approaches and asymptotic approaches.

* Corresponding author. Fax: +33 1 64153741.
E-mail addresses: rawad@rawadbaroud.com (R. Baroud), karam.sab@enpc.fr (K. Sab), jean-francois.caron@enpc.fr (J.-F. Caron), fouad.kaddah@usj.edu.lb (F. Kaddah).

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In asymptotic approaches, a plate model is derived directly from the full 3D formulation of the problem, assuming the thickness of the plate goes to zero. In these approaches, the asymptotic expansion method plays a central role and the leading order leads to Kirchhoff–Love plate theory (Ciarlet and Destuynder, 1979). In axiomatic approaches, 3D fields are assumed a priori and a plate theory is derived using integration through the thickness and variational tools (Altenbach, 1998; Kienzler, 2002; Touratier, 1992; Vekua, 1985). Recently, Lebée and Sab (2011a, 2011b, 2012) have proposed a new plate theory for the analysis of thick plates under out-of-plane loading. This theory, called the Bending-Gradient plate theory, is an extension of the Reissner–Mindlin plate theory and improves the predictions of shear stress distributions in laminated plates. Although ESL models can provide acceptable results for global response of multilayers, they may lead to very inaccurate estimations of local response especially near free-edges.

Layerwise models have been proposed to overcome the drawbacks of ESL models (Barbero and Reddy, 1991; Botello et al., 1999; Carrera, 1998; Dakshina Moorthya and Reddy, 1998; Gaudenzi et al., 1995; Robbins and Reddy, 1993). In these approaches, each layer of the multilayered structure is considered as an independent plate. Therefore, the number of governing equations depends on the number of the layers. This increases significantly the computational cost in layerwise approaches. However, thanks to their accuracy with respect to ESL models and their efficiency with respect to full 3D models, layerwise models have been proven to be very good alternatives to 3D models. The interested reader car refer to Carrera (2002, 2004); Zhang and Yang (2009) for reviews of different theories.

By taking a direct inspiration from Pagano's model (Pagano, 1978), a layerwise stress model was proposed in Caron et al. (2006); Carreira et al. (2002); Dallot and Sab (2008); Diaz Diaz et al. (2002); Lerpiniere et al. (2014); Naciri et al. (1998); Nguyen and Caron (2006); Saeedi et al. (2012a, 2012b, 2013a); 2013b). In this model, the multilayered material is considered as a superposition of Reissner–Mindlin plates linked together by interfacial stresses which are considered as additional generalized stresses. In order to make reference to Carrera's nomenclature proposed in Carrera (2004), this model, previously called *Multi-particle Model of Multilayered Materials (M4)*, was renamed as *LS1* which means a Layerwise Stress approach with first-order membrane stress approximations per layer in the thickness direction.

The main difference between the *LS1* model and other existing layerwise models is that, most often, the other layerwise models are either displacement approaches or mixed displacement-stress approaches while the *LS1* model, derived by means of the Hellinger–Reissner mixed principle, is a pure layerwise stress approach where there are no assumptions on the displacement fields. Diaz Diaz et al. (2002) used the *LS1* model to evaluate interfacial stresses in symmetrical laminates under tensile loading with free edges. Caron et al. (2006) applied this model to the prediction of mode III delamination in multi-layered materials. In Dallot and Sab (2008), the authors employed the *LS1* model for analyzing a sandwich plate under cylindrical bending and demonstrated the capacity of this model to capture the plastic collapse modes. In Saeedi et al. (2012b), the authors proposed the *refined LS1* model by introducing several *mathematical layers* per physical layer in order to capture the stress concentrations occurring in delaminated multilayered plates under uniaxial tension. It was proved that the proposed layerwise mesh strategy improves considerably stress and energy release rate estimations given by the non refined *LS1* model considered in Saeedi et al. (2012a). Nevertheless, the non refined *LS1* model shows itself very effective in the simulation of mode I (Double Cantilever) and mode II (End Notched Flexure) delamination tests on multilayered plates (Saeedi et al., 2013a), and in the simulation of delamination propagation in multilayered materials at $0°/\theta°$ interfaces (Lerpiniere et al., 2014).

Even if the *LS1* model and its refined version are very effective models, they can be still improved. Indeed, first, the 3D stress free boundary conditions cannot be exactly met by these models, and second, as these models are derived by means of the Hellinger–Reissner mixed variational principle, there is no theoretical guarantee of the convergence of the *refined LS1* model to the 3D model, as the number of mathematical layers per physical layer increases.

The objective of this paper is to improve the *LS1* model by removing these drawbacks. For this purpose, a new layerwise model, called *Statically Compatible LS1 (SCLS1)*, is introduced. As in *LS1*, the laminated plate is still considered as a superposition of Reissner plates coupled by interfacial stresses. However, the divergences of the interlaminar transverse shears are introduced as new generalized efforts. Moreover, the new model is derived by means of the minimum of the complementary potential energy ensuring the convergence of its refined version to the exact 3D model, as the number of mathematical layers per physical layer increases.

Furthermore, the new model has been implemented in a new version of the in-house finite element code called *MPFEAP (for MultiParticle Finite Element Analysis Program)*. The new 2D finite element has eight nodes with $6n-1$ degrees of freedom per node, n being the number of layers constituting the laminate. Finally, several comparisons are made between *LS1, SCLS1, refined LS1, refined SCLS1* and full 3D FE models for straight free edge plate and notched laminate under uniaxial tension.

The paper is organized as follows: The next section is dedicated to the theoretical formulation of the *SCLS1* model referring also to the original *LS1* model, highlighting the differences. Section 3 discusses the FE discretization of the new model and its implementation. Section 4 presents the numerical comparisons between the different models. The paper ends with a conclusion which synthesizes the main results and discusses prospects for future developments.

2. Theoretical formulation of the statically compatible model *SCLS1*

In this section, a new model for linear elastic multilayered plates called *SCLS1* is described. The *SCLS1* model is derived from the 3D exact model by considering Statically Compatible Layerwise Stresses with first-order membrane stress approximations per layer in the thickness direction. The generalized stresses of the proposed model are actually those of a Reissner–Mindlin plate per layer in addition to inter-laminar shear and normal stresses at the interfaces between layers and the divergences of these inter-laminar shear stresses. The exact 3D equilibrium equations lead to $6n-1$ equilibrium equations on the generalized stresses, where n is the number of layers. Therefore, the kinematics of the *SCLS1* model, obtained by dualization of the equilibrium equations, has $6n-1$ degrees of freedom at each point of the middle surface of the plate. Finally, the generalized constitutive equations of the *SCLS1* model linking the generalized stresses to the generalized strains are derived by using the stress energy formulation.

2.1. Problem description and notations

The multilayered plate under consideration is composed of n perfectly bonded orthotropic elastic layers (Fig. 1). The plate occupies the 3D domain $\Omega = \omega \times]h_1^-, h_n^+[$ where ω is the middle surface of the plate. In the following, x and y are the in-plane coordinates and z is the out-of-plane coordinate. The following notations are introduced:

- The superscripts i and $j, j+1$ indicate layer i and the interface between layer j and j+1 with $1 \leq i \leq n$ and $1 \leqslant j \leqslant n-1$, re-