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Analytic determination of stress fields in cross-ply symmetric composite laminates

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ABSTRACT

A Hamiltonian state space approach for analytic determination of deformation and stress fields in multilayered cross-ply symmetric laminates under extension and bending is presented, in which the equations of anisotropic elasticity, the end conditions, the traction-free boundary conditions on the bounding planes of the rectangular section, and the interfacial continuity conditions in multilayered laminates are satisfied, regardless of the number of layers. The solutions serve as useful benchmarks for numerical modeling and material characterization of composite laminates.

1. Introduction

This study develops an effective and systematic approach for analytic determination of deformation and stress fields in multilayered composite laminates, which is important in experimentally determining the lamina properties and verifying numerical models. Although this class of problems has been studied for many years (Aghdam and Falahatgar, 2003; Andakhshideh and Tahani, 2013; Becker, 1993; Chang and Tarn, 2007; Cho and Kim, 2000; Dong and Goetschel, 1982; Han et al., 2014; Kassapoglou and Lagace, 1987; Tahani and Asghar, 2003; Zhang et al., 2006), with works proposing both numerical and analytical solutions based on various approximations and simplifications, to the best of our knowledge, there are few exact analyses for extension and bending of composite laminates in the literature. Analyses of these problems using the conventional approach find it difficult to satisfy the interfacial continuity conditions and the edge boundary conditions. It is thus, often necessary to relax either the exact edge boundary conditions or the interfacial continuity conditions or both. For example, on the basis of the classical lamination theory (CLT) and various higher-order theories (Christensen, 1979; Jones, 1975) such as the first-order shear deformation (FSDT) and the higher-order shear deformation theory (HSDT), the through-thickness variation of the displacement is assumed, and they did not discuss the interfacial continuity conditions and the traction-free boundary conditions at the free edges. The traction-free boundary conditions along the edges require the stresses vanish point by point on the edge surfaces, and these conditions are difficult to satisfy exactly. One must then resort to Saint-Venant's principle to replace the edge conditions by equivalent ones, such that the stress resultants across the thickness are equal to zero at the free edge. The solutions thus obtained are considered to be valid away from the edge surfaces (Mian and Spencer, 1998; Rogers et al., 1992; Tarn and Huang, 2002; Wang et al., 2000).

An exact analysis is possible only for limited cases. Pagano (1969, 1970) presented the well-known analytic solutions on bending of simply-supported symmetric cross-ply laminates. Wang and Choi (1982a, 1982b) determine the order of stress singularities near the free edges using Lekhnitskii's stress potentials and assuming power-law for the stresses. The particular solution of the governing equations was obtained by assuming a cubic polynomial, whereas the homogeneous solution was expanded as an infinite series of eigenfunctions. Free parameters in the eigenfunctions were then computed by a boundary collocation method to satisfy the remote boundary conditions. Ren (1987) presented exact solutions in terms of Fourier series and an Airy stress function for laminated cylindrical shells in cylindrical bending under plane strain conditions. Noor and Burton (1990) expressed the six stress components and the three displacement components of the three-dimensional elasticity in terms of a double Fourier series in the Cartesian coordinates, and presented the stress and free vibration problems of simply-supported antisymmetric laminate.

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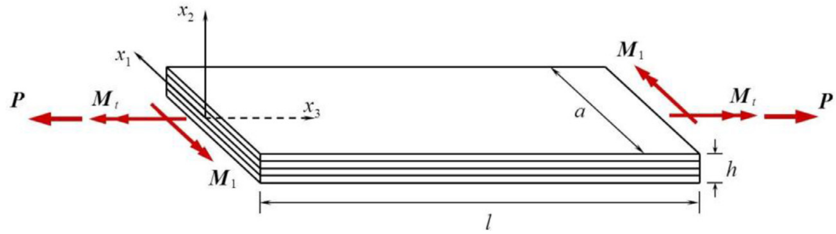

Fig. 1. The free-edge problem for a composite laminate of rectangular section subjected to an axial force P, a bending moment M_1 and a torque M_t at the end sections.

Savoia and Reddy (1992) assumed a displacement field and solved the three-dimensional plate problem by minimizing the total potential energy functional, and solutions for cross-ply and antisymmetric angle-ply laminated plates had been presented. Heyliger (1997) presented some exact solutions for the static behavior of piezoelectric laminate with simply support subjected to transverse loadings. Batra and Liang (1997) used the three-dimensional linear elasticity to analyze the steady-state vibrations of a simply-supported rectangular laminated plate with embedded PZT layers. Werner (1999) presented a Navier-type three-dimensional exact solution for small deflections in bending of linear elastic isotropic homogeneous rectangular plates. Pan (2001) derived exact solutions for three-dimensional, linearly magneto-electro-elastic, and rectangular plates under static loadings. The solutions are expressed in terms of the propagator matrix and include all the previous solutions. Meyer-Piening (2004) presented a sandwich plate bending analysis on the basis of the three-dimensional elasticity. Zenkour (2007) analyzed the bending problem of symmetric and antisymmetric simply supported cross-ply laminates using the three-dimensional elasticity and the state space concept. Williams (1999) developed a multiscale theory for analyzing the history-dependent response of laminated plates. Demasi (2007, 2010) developed a Navier-type method by using the mixed form of Hooke's Law which leads to formulating the boundary conditions on the top and bottom surfaces of the plate directly in terms of transverse stresses, and presented an exact three-dimensional solution for isotropic and orthotropic simply-supported rectangular plates. In the above cited solutions, the three-dimensional linear theory of elasticity are used, and the components of displacement, stress, and electric displacement are expressed in the form of a double Fourier series, so that the solutions are mainly suitable for the case of simply-supported boundary conditions of laminated rectangular plates subjected to a sinusoidally/uniformly distributed load.

Using the theory of anisotropic elasticity, Tarn and Chang (2013) formulated the basic equations of anisotropic elasticity and piezoelasticity into the state space framework by a Hamiltonian variation formulation. Based on the Hamiltonian state space formalism, the present study derives the analytic solutions for extension and bending of symmetric cross-ply laminates without a priori assumptions. Guided by previous study (Liang et al., 2014), we seek the eigensolutions in the form of exponential functions of the distance from the edges in the finite cross section of the plate, where the transfer matrix is employed to satisfy the interfacial continuity and boundary conditions. Regardless of the number of layers, the approach requires only a systematic operation of 4 × 4 matrices. The solutions obtained herein are exact in that the displacement and stress fields satisfy the basic equations of anisotropic elasticity, the free-edge boundary conditions, the traction-free boundary conditions on the top and bottom planes, the end conditions, and the interfacial continuity conditions in multilayered composite laminates. Comparisons of the stress fields between the proposed analytic and finite element solutions show good agreement. The presented elasticity solutions are important because they can be used to study the boundary-layer effects of composite laminates, in addition to serving as benchmarks for the evaluation of numerical solutions for the layered composite structures.

The main process of the present approach can be organized as follows:

1. Formulate the state space equations on the basis of the three-dimensional elasticity.
2. Determine a particular solution which satisfies the nonhomogeneous state space equations, the interfacial continuity conditions, and the boundary conditions on the top and bottom planes.
3. Determine an eigensolution which satisfies the homogeneous term of the state space equations, the interfacial continuity conditions, and the boundary conditions on the top and bottom planes.
4. Add the eigensolution to the particular solution, and satisfying the free-edge conditions on the left and right planes by using the symplectic orthogonality of the eigenvector.

2. Problem statement

Consider a symmetric composite laminate composed of m anisotropic elastic layers of the rectangular section subjected to an axial force, a bending moment and a torque at the end sections, as shown in Fig. 1. Using Cartesian coordinates (x_1, x_2, x_3), the origin is located at the center of the middle plane. The x_2-axis is pointing in the thickness direction such that the top and bottom planes and the interfaces between adjacent layers are defined by $x_2 = constant$. The top and bottom planes, the free-edge boundaries, and the end sections are defined by $x_2 = \pm h/2,\ x_1 = \pm a/2$ and $x_3 = 0,\ l$, respectively. Following Chang and Tarn (2007), the three-dimensional equations of the elasticity can be expressed in matrix forms as follows.

Generalized Hooke's law:

$$\begin{bmatrix} \boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_2 \\ \boldsymbol{\sigma}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} & \mathbf{C}_{13} \\ \mathbf{C}_{21} & \mathbf{C}_{22} & \mathbf{C}_{23} \\ \mathbf{C}_{31} & \mathbf{C}_{32} & \mathbf{C}_{33} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon}_1 \\ \boldsymbol{\varepsilon}_2 \\ \boldsymbol{\varepsilon}_3 \end{bmatrix}, \tag{1}$$

where σ_i denotes the stress vector in the x_i direction, ε_i consists of the corresponding strain components, and C_{ij} are matrices of the elastic constants given below,

$$\boldsymbol{\sigma}_i = [\sigma_{1i} \quad \sigma_{2i} \quad \sigma_{3i}]^T, \quad \boldsymbol{\varepsilon}_i = [\varepsilon_{1i} \quad \varepsilon_{2i} \quad \varepsilon_{3i}]^T,$$

$$\mathbf{C}_{21} = \mathbf{C}_{12}^T, \quad \mathbf{C}_{31} = \mathbf{C}_{13}^T, \quad \mathbf{C}_{32} = \mathbf{C}_{23}^T,$$

$$\mathbf{C}_{11} = \begin{bmatrix} c_{11} & c_{16} & c_{15} \\ c_{16} & c_{66} & c_{56} \\ c_{15} & c_{56} & c_{55} \end{bmatrix}, \quad \mathbf{C}_{12} = \begin{bmatrix} c_{16} & c_{12} & c_{14} \\ c_{66} & c_{26} & c_{46} \\ c_{56} & c_{25} & c_{45} \end{bmatrix},$$

$$\mathbf{C}_{13} = \begin{bmatrix} c_{15} & c_{14} & c_{13} \\ c_{56} & c_{46} & c_{36} \\ c_{55} & c_{45} & c_{35} \end{bmatrix}, \quad \mathbf{C}_{22} = \begin{bmatrix} c_{66} & c_{26} & c_{46} \\ c_{26} & c_{22} & c_{24} \\ c_{46} & c_{24} & c_{44} \end{bmatrix},$$

$$\mathbf{C}_{23} = \begin{bmatrix} c_{56} & c_{46} & c_{36} \\ c_{25} & c_{24} & c_{23} \\ c_{45} & c_{44} & c_{34} \end{bmatrix}, \quad \mathbf{C}_{33} = \begin{bmatrix} c_{55} & c_{45} & c_{35} \\ c_{45} & c_{44} & c_{34} \\ c_{35} & c_{34} & c_{33} \end{bmatrix}.$$