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# Stochastic stability of a fractional viscoelastic column under bounded noise excitation

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ABSTRACT

The stability of a viscoelastic column under the excitation of stochastic axial compressive load is investigated in this paper. The material of the column is modeled using a fractional Kelvin–Voigt constitutive relation, which leads to that the equation of motion is governed by a stochastic fractional equation with parametric excitation. The excitation is modeled as a bounded noise, which is a realistic model of stochastic fluctuation in engineering applications. The method of stochastic averaging is used to approximate the responses of the original dynamical system by a new set of averaged variables which are diffusive Markov vector. An eigenvalue problem is formulated from the averaged equations, from which the moment Lyapunov exponent is determined for the column system with small damping and weak excitation. The effects of various parameters on the stochastic stability and significant parametric resonance are discussed and confirmed by simulation results.


## 1. Introduction

Stochastic dynamic analysis is extensively used in civil and mining engineering, as the loadings imposed on the structures are quite often of random nature, such as those arising from earthquakes, wind, explosion, and ocean waves, which can be characterized satisfactorily only in probabilistic terms. The dynamic responses of these engineering structures are governed in general by $n$-dimensional stochastic differential equations of the form

$$\dot{X}_j = f_j(t, \mathbf{X}, \boldsymbol{\xi}), \quad j = 1, 2, \ldots, n, \tag{1}$$

where $\mathbf{X} = \{X_1, X_2, \ldots, X_n\}^T$ is the state vector of the system and $\boldsymbol{\xi}$ is a vector of stochastic loadings. For engineering applications, the stochastic loadings have been modeled as a Gaussian white noise process, a real noise process, or a bounded noise process.

A white noise process is a weakly stationary process that is delta-correlated and mean zero. Its power spectral density is constant over the entire frequency range, which is obviously an idealization. A real noise $\xi(t)$ is often characterized by an Ornstein–Uhlenbeck process and is given by $d\xi(t) = -\alpha\xi(t)dt + \sigma\, dW(t)$, where $W(t)$ is a standard Wiener process. It is well known that $\xi(t)$ is a normally distributed random variable, which is not bounded and may take arbitrarily large values with small probabilities, and hence may not be a realistic model of noise in many engineering applications.

A bounded noise $\xi(t)$ is normally represented as

$$\xi(t) = \zeta \cos[\nu t + \sigma W(t) + \theta], \tag{2}$$

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where $\zeta$ is the noise amplitude, $\sigma$ is the noise intensity, $W(t)$ is the standard Wiener process and $\theta$ is a random variable uniformly distributed in the interval $[0, 2\pi]$. The inclusion of the phase angle $\theta$ makes the bounded noise $\xi(t)$ a stationary process. Eq. (2) may be written as

$$\xi(t) = \zeta \cos Z(t), \quad \mathrm{d}Z(t) = \nu t + \sigma \circ \mathrm{d}W(t), \tag{3}$$

where the initial condition of $Z(t)$ is $Z(0) = \theta$. The auto-correlation function of $\xi(t)$ is given by

$$R(\tau) = \mathrm{E}[\xi(t)\xi(t+\tau)] = \frac{1}{2}\zeta^2 \cos \nu\tau \exp\left(-\frac{\sigma^2}{2}|\tau|\right), \tag{4}$$

and the spectral density function of $\xi(t)$ is

$$S(\omega) = \int_{-\infty}^{+\infty} R(\tau)\mathrm{e}^{-\mathrm{i}\omega\tau}\,\mathrm{d}\tau = \frac{\zeta^2\sigma^2\left(\omega^2 + \nu^2 + \frac{1}{4}\sigma^4\right)}{2\left[(\omega+\nu)^2 + \frac{1}{4}\sigma^4\right]\left[(\omega-\nu)^2 + \frac{1}{4}\sigma^4\right]}. \tag{5}$$

Furthermore, $|\xi(t)| < \zeta$ for all values of the time $t$ and hence is a bounded stochastic process. When the parameter $\sigma$ in $\xi(t)$ is small, the bounded noise can be used to model a narrow-band process about frequency $\nu$. In the limit as $\sigma$ approaches zero, the bounded noise reduces to a deterministic sinusoidal function. On the other hand, in the limit as $\sigma$ approaches infinite, the bounded noise becomes a white noise of constant spectral density. However, since the mean-square value is fixed at $\frac{1}{2}$, this constant spectral density level reduces to zero in the limit. The bounded noise process was first employed by Stratonovich [1] and has since been applied in certain engineering applications [2–5].

One of the most important dynamical properties of the solution of stochastic system (1) is its dynamic stability. The sample or almost sure stability of system (1) is governed by the Lyapunov exponents defined as

$$\lambda_{\mathbf{X}} = \lim_{t\to\infty} \frac{1}{t} \log \|\mathbf{X}\|, \tag{6}$$

where $\|\mathbf{X}\| = (\mathbf{X}^T\mathbf{X})^{1/2}$ is the Euclidean norm. If the largest Lyapunov exponent is negative, the trivial solution of system (1) is stable with probability 1; otherwise, it is unstable almost surely. Lyapunov exponents characterize sample stability or instability. However, this sample stability cannot assure the moment stability. Hence, to obtain a complete picture of the dynamic stability, it is important to study both the top Lyapunov exponent and the moment Lyapunov exponent.

The stability of the $p$th moment $\mathrm{E}[\|\mathbf{X}\|^p]$ of the solution of system (1) is governed by the $p$th moment Lyapunov exponent defined by

$$\Lambda_{\mathbf{X}}(p) = \lim_{t\to\infty} \frac{1}{t} \log \mathrm{E}\left[\|\mathbf{X}\|^p\right], \tag{7}$$

where $\mathrm{E}[\cdot]$ denotes the expected value. If $\Lambda_{\mathbf{X}}(p)$ is negative, then the $p$th moment is stable; otherwise, it is unstable. The $p$th moment Lyapunov exponent $\Lambda_{\mathbf{X}}(p)$ is a convex analytic function in $p$ that passes through the origin and the slope at the origin is equal to the largest Lyapunov exponent $\lambda_{\mathbf{X}}$, i.e.

$$\lambda_{\mathbf{X}} = \lim_{p\to 0} \frac{\partial \Lambda_{\mathbf{X}}(p)}{\partial p}. \tag{8}$$

The non-trivial zero $\delta_{\mathbf{X}}$ of $\Lambda_{\mathbf{X}}(p)$, i.e. $\Lambda_{\mathbf{X}}(\delta_{\mathbf{X}}) = 0$, is called the stability index.

The increasing use of materials such as polymers, composite materials, metals, rocks, and alloys at elevated temperatures has emphasized the need for development of theories for analyzing viscoelastic structures under dynamic loadings. The dynamic stability of viscoelastic systems has been investigated by some authors [6]. The equation of motion of the viscoelastic system under stochastic excitations is usually governed by the stochastic integro-differential equation and the response and stability of the system is difficult to be obtained exactly. Therefore, several numerical and approximate procedures have been proposed. Potapov described the behavior of stochastic viscoelastic systems by numerical evaluation of Lyapunov exponents of linear integro-differential equations [7], and he studied the almost-sure stability of a viscoelastic column under the excitation of a random wide band stationary process using Lyapunov's direct method [8]. The method of stochastic averaging, originally formulated by Stratonovich [1] and mathematically proved by Khasminskii [9], has been widely used to approximately solve stochastic differential equations containing a small parameter. Another reason is that under certain conditions stochastic averaging can reduce the dimension of problems concerned to one dimension, and then greatly simplify the solution [5].

An increasing interest has been directed to non-integer or fractional viscoelastic constitutive models [10]. In contrast to the well-established mechanical models based on Hookean springs and Newtonian dashpots, which results in an exponential decay of the relaxation function, the fractional models accommodate non-exponential relaxation, which makes it possible to model hereditary property with long memory. Fractional constitutive models lead asymptotically to power law behavior in linear viscoelasticity [11]. There is a theoretical reason for using fractional calculus in viscoelasticity, in which the molecular theory of Rouse [12] gives relationship between stress and strain with fractional derivative of strain. Experiments also revealed that the viscous damping behavior can be described in an excellent way by the introduction of fractional derivatives in stress–strain relations [13,14].