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Revisiting a magneto-elastic strange attractor

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ABSTRACT

We revisit an early example of a nonlinear oscillator that exhibits chaotic motions when subjected to periodic excitation: the magneto-elastically buckled beam. In the paper of Moons and Holmes (1980) [1] magnetic field calculations were outlined but not carried through; instead the nonlinear forces responsible for creation of a two-well potential and buckling were fitted to a polynomial function after reduction to a single mode model. In the present paper we compute the full magnetic field and use it to approximate the forces acting on the beam, also using a single mode reduction. This provides a complete model that accurately predicts equilibria, bifurcations, and free oscillation frequencies of an experimental device. We also compare some periodic, transient and chaotic motions with those obtained by numerical simulations of the single mode model, further illustrating the rich dynamical behavior of this simple electromechanical system.

1. Introduction

In this paper we return to a classical problem in nonlinear vibrations: the single degree-of-freedom (dof) Düffing's equation [2] with an unstable equilibrium flanked by two stable equilibria. Unlike the stiffening spring case considered in many textbooks, e.g. [3], the two-well potential energy of this system endows its phase space with a homoclinic orbit [4] which undergoes global bifurcations that produce non-periodic, chaotic motions when external periodic forcing is applied. This mathematical model with its physical analog of an elastic cantilever beam buckled by ferromagnetic forces, exemplified in Fig. 1, has become a central example in “chaos theory” since its introduction in 1979 [1], e.g. see [5,6].

The simplest model for a single dof oscillator with a two-well potential has a symmetric cubic restoring force and linear damping. With additive sinusoidal forcing, the ODE takes the form

$$\ddot{x}+\delta\dot{x}-\alpha x+\beta x^3=P\cos(\omega t), \tag{1}$$

where the parameters $\alpha,\ \beta,\ \omega>0$, and $\delta,\ P\geq 0$. When $P=\delta=0$, Eq. (1) is a Hamiltonian system that conserves energy

$$H(x,\dot{x})=\frac{\dot{x}^2}{2}-\frac{\alpha x^2}{2}+\frac{\beta x^4}{4}, \tag{2}$$

so that orbits of Eq. (1) lie on level sets of $H(x,\dot{x})$ [4] and we observe a double homoclinic loop containing a saddle point on the set $H(x,\dot{x})=0$. Perturbing from this integrable case, it was proved in [7] that, for fixed $\alpha,\ \beta,\ \omega>0$, $0<\varepsilon\ll 1$, $\delta, P=\mathcal{O}(\varepsilon)$ and $P>\delta P_c(\alpha,\beta,\omega)$ (a critical value, see Eq. (14) below), Eq. (1) has transverse homoclinic orbits and therefore possesses a chaotic invariant set containing infinitely many unstable periodic orbits: a Smale horseshoe [4, Sections 4.5 and 5.1]. An analogous result was subsequently proved for a PDE modeling vibrations of a simply supported buckled beam [8] and there have been numerous similar studies of systems with 2 or more degrees of freedom, e.g. [9–11].

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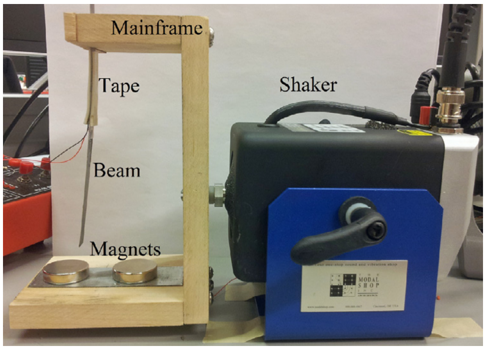

Fig. 1. The magneto-elastic beam and shaker. The frame measures $153 \times 100 \times 76$ mm^3.

These results do not imply that Eq. (1) has a strange attractor [4] (indeed, the stable manifolds of the horseshoe may form a fractal basin boundary separating the domains of attraction of two stable periodic orbits that "grow" from the stable equilibria of (1) for $P=0$ [6,4]), but many subsequent studies and analyses of Poincaré maps strongly suggest that there are open (or at least measurable) sets in parameter space for which strange attractors do exist. For an introductory article with animations, see [12].

The present paper revisits the analysis of [1], augmenting it with explicit numerical calculations of the magnetic field, from which the nonlinear stiffness term in the single dof model may be derived explicitly. We compare the resulting function with the cubic of Eq. (1), fitting the parameters α and β from experimental observations of equilibrium positions and natural frequencies and deriving the damping factor δ from free vibration decay rates. We find that the model calculations predict key quantities within 10 percent, and that bifurcation diagrams also match experimental results well, including transitions from two to three stable equilibria and symmetry-breaking as magnet positions vary with respect to the clamped end of the beam.

This paper is organized as follows. We first describe the magneto-elastic beam used for experimental data collection in Section 2, since this simple physical system also underlies the model. In Section 3 we review the partial differential equation (PDE) describing the beam and outline its reduction by Galerkin's method to an ordinary differential equation (ODE) describing the fundamental flexural mode. The analysis follows [1], supplemented by a recent efficient algorithm due to Derby and Olbert [13] that is used to compute the magnetic field and resulting restoring forces. Polynomial approximations to the restoring forces are also derived, analogous to those considered in [1]. In Section 4 we compare predictions of equilibrium positions and natural frequencies derived from the single mode model of Section 3 with experimental measurements, and illustrate bifurcations of equilibria that occur under changes in the magnetic field due to magnet placement relative to the beam's support. The dynamics of periodically forced vibrations are considered in Section 5, including subharmonics and non-periodic, chaotic motions, which are studied using Poincaré maps. Conclusions and a brief discussion follow in Section 6.

2. The experimental rig

The physical system, shown in Fig. 1, is a slender steel cantilever beam clamped in a nominally rigid frame at its upper end and free at its lower end. The beam moves under the influence of elastic, magnetic, and noninertial forces (when the frame is excited horizontally by the shaker). Gravitational forces also act, but these are much smaller than the forces noted above and will be neglected in the model developed in Section 3.

The frame is fabricated in 12.3 mm thick maple hardwood with the grain oriented longitudinally in each component. Components are joined by 3 woodscrews at top and bottom and a strip of 1018 steel is glued to the upper surface of the base for magnet attachment. The beam is 1095 blue tempered spring steel of Young's modulus $E=2.06 \times 10^5$ MPa, density $\rho=(7.83 \pm 0.02) \times 10^3$ kg m^{-3}, with thickness $\Delta=0.25 \pm 0.02$ mm and width $w=9.5 \pm 0.5$ mm, clamped at the top of the frame by a maple strip secured by 2 woodscrews. To increase damping, strips of Scotch tape 39.2 ± 0.1 mm long, 9.6 ± 0.1 mm wide and 1.5 ± 0.2 mm thick are attached to both sides of the beam near the clamp. Experiments were conducted with free beam length $L=108.8 \pm 0.5$ mm below the clamp and distance from beam tip to frame base 19.2 ± 0.5 mm. Beam dimensions were chosen so that the fundamental flexural mode is well separated in frequency from torsional and higher flexural modes.

Two cylindrical rare earth magnets of radius $r=12.7$ mm, height $h=6.35$ mm and surface field strength $B_{surface}=0.21$ Tesla are placed with matching N–S polarities on the steel strip, whose upper surface is roughened to prevent magnet movement. Magnet center positions can be varied upward from a minimum separation of 37.4 ± 0.1 mm. Most of the experiments are done with magnet centers (approximately) equally spaced from the undeflected beam tip, but asymmetric cases are also considered. The frame is attached to the driving shaft of the shaker by a machine screw whose axis passes through the frame's mass center, as computed from the assembly with magnets as modeled in Creo Pro/Engineer. Sinusoidal excitation is provided by a K2007E01 SmartShaker electromagnetic shaker with an integrated amplifier (The Modal Shop,