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Buckling and Vibration of Functionally Graded Material Columns Sharing Duncan's Mode Shape, and New Cases

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ABSTRACT

In this study, the closed-form solution for the buckling of an inhomogeneous simply supported column that was uncovered by the noted British engineer Duncan in 1937, is first derived in a straightforward manner. It deals with buckling of a centrally compressed inhomogeneous column. It is also found that there are several other columns with variable axial functionally graded material (FGM) that share the same qualities as Duncan's column. It is then shown that the mode postulated by W.J. Duncan (1894–1970), FRS and the newly found modes, have a greater validity, namely the freely vibrating beam, albeit with different flexural rigidity than the centrally compressed one, may possess the same buckling mode. It is demonstrated also that there exists an inhomogeneous beam under axial compression whose vibration mode coincides with the buckling modes in the previous cases.

1. Introduction

Duncan [3] devoted his study to efficacy of the Bubnov-Galerkin method. *Inter alia*, he communicated, without derivation, closed-form solution for buckling of inhomogeneous columns. As is known, the closed form solutions for inhomogeneous structures are extremely rare. Therefore, it is interesting to know how Duncan obtained his solution. Moreover, a pertinent question arises if there are other columns or beams for which Duncan's mode shape is valid, or if there are other similar examples.

This study addresses above issues. It shows how one can derive Duncan's classic solution, and constructs analogous solutions for the vibration problems. Remarkably, it turns out, that there exists a vibrating column whose vibration mode coincides with Duncan's buckling mode. Note that monograph by Elishakoff [5] contains analysis for other candidate mode shapes of beams in vibratory or buckling conditions. The present study is apparently the first one that addresses Duncan's mode shape directly.

Duncan [3] proposed that the shape of the mode be taken as

$$W(\xi) = 7\xi - 10\xi^3 + 3\xi^5 \quad (1)$$

and this shape satisfies the simple support conditions at the two ends. Later on, Elishakoff [5] suggested another mode

$$W(\xi) = \xi - 2\xi^3 + \xi^4 \quad (2)$$

which has similar properties. These two case can be realized with spatial distribution of material properties that will be given below. The question that one may ask is if there exist other simple shapes that have similar properties. These new cases will yield different buckling loads and spatial distribution of the material properties. In the next section a general derivation is presented for the problem. Other recent studies of inhomogeneous beams and columns include those of Akulenko and Nesterov [1], Caruntu [2], Ece, Ayadoğlu and Taskin [4], Sina and Navazi [10], Gilat, Caliò and Elishakoff [6], Huang and Li [7], Huang and Luo [8], Zarrinzadeh, Attarnejad and Shahba [11], and Maròti [9], among others.

2. Derivation of Duncan's solution and other new solutions

Consider the governing differential equation for the buckling of centrally compressed inhomogeneous column simply supported at its two ends:

$$D(\xi)\frac{d^2W}{d\xi^2} + P_{cr}L^2W = 0. \quad (3)$$

One can show that the function in Eq. (1), postulated by Duncan [3] satisfies the boundary conditions of the simple supports

$$W(0) = D(\xi)W''(0) = W(1) = D(\xi)W''(1) = 0 \quad (4)$$

where the prime denotes the differentiation with respect to ξ. We pose the following question: Is there an inhomogeneous column that has expression in Eq. (1) as its buckling mode? To answer this question, we observe that the second term in Eq. (1), namely, $P_{cr}L^2w$ represents a

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fifth order polynomial. The second derivative of the buckling mode in the first term is a third order polynomial. Therefore, in order for the first term $D(\xi)w''$ to be also a fifth order polynomial, it is sufficient that $D(\xi)$ is a second order polynomial. Hence, we seek $D(\xi)$in the following form

$$D(\xi) = d_0 + d_1\xi + d_2\xi^2. \tag{5}$$

Now we look for other possible fifth order polynomials that can be the mode shape for buckling of a simply supported beam. Then we shall try the following mode

$$W(\xi) = w_0 + w_1\xi + w_2\xi^2 + w_3\xi^3 + w_4\xi^4 + w_5\xi^5. \tag{6}$$

Since we have a simple support at $\xi = 0$ we must have $w_0 = w_2 = 0$. Then we substitute Eqs. (5) and (6) into Eq. (3) and collect terms with the same power of ξ and obtain the following five equations:

$$6w_3 + PL^2w_1 = 0 \tag{7}$$

$$12w_4 + 6w_3d_1 = 0 \tag{8}$$

$$6w_3d_2 + 12w_4d_1 + 20w_5 + PL^2w_3 = 0 \tag{9}$$

$$20w_5d_1 + 12w_4d_2 + PL^2w_4 = 0 \tag{10}$$

$$20w_5d_2 + PL^2w_5 = 0 \tag{11}$$

and two more equations are obtained from the requirement of zero deflection and moment at $\xi = 1$ as

$$w_1 + w_3 + w_4 + w_5 = 0, \tag{12}$$

$$6w_3 + 12w_4 + 20w_5 = 0. \tag{13}$$

Eqs. ((7)–(13)) represent a set of 7 nonlinear equations with seven unknowns $w_1, w_3, w_4, w_5, d_1, d_2,$ and P. We obtain four solutions (and one trivial solution where all the unknowns are zero).

a. First solution—Duncan's [3] mode shape

$$W(\xi) = 7\xi - 10\xi^3 + 3\xi^5; \quad D(\xi) = d_0\left(1 - \frac{3}{7}\xi^2\right); \quad P_{cr} = \frac{60d_0}{7L^2}. \tag{14}$$

b. Second solution—Elishakoff's [5] mode shape

$$W(\xi) = \xi - 2\xi^3 + \xi^4; \quad D(\xi) = d_0\left(1 + \xi - \xi^2\right); \quad P_{cr} = \frac{12d_0}{L^2}. \tag{15}$$

c. Third solution—First new mode shape

$$W(\xi) = \frac{8}{15}\xi - \frac{4}{3}\xi^3 + \xi^4 - \frac{1}{5}\xi^5; \quad D(\xi) = d_0\left(1 + \frac{3}{2}\xi - \frac{3}{4}\xi^2\right); \quad P_{cr} = \frac{15d_0}{L^2}. \tag{16}$$

d. Fourth solution—Second new mode shape

$$W(\xi) = \frac{1}{15}\xi - \frac{2}{3}\xi^3 + \xi^4 - \frac{2}{5}\xi^5; \quad D(\xi) = d_0\left(1 + 3\xi - 3\xi^2\right); \quad P_{cr} = \frac{60d_0}{L^2}. \tag{17}$$

These four solutions are listed in Table 1 with the modes and the corresponding stiffness distribution along the beam. It is evident that solution (d) above is the second buckling mode as can be also seen from the value of the buckling load that is much higher than the value of the three other solutions. Additionally, the associated mode shape possesses an internal node, serving as an indication that one deals with the second mode-shape.

3. Comparison with uniform column

Let us compare the derived buckling load with that of the associated uniform column. We can introduce the latter columns as that with average flexural rigidity, defined as

$$D_{ave} = \int_0^1 D(\xi)d\xi. \tag{18}$$

In the Duncan's [3] example, the average flexural rigidity, in view of Eq. (14) is

$$D_{ave} = \frac{6}{7}d_0. \tag{19}$$

Thus, $d_0 = 7/6D_{ave}$. The buckling load is from Eq. (14)

$$P_{cr} = \frac{10D_{ave}}{L^2} \tag{20}$$

which is extremely close, from above, to the Euler buckling load of the *uniform* column with flexural rigidity D_{ave}:

$$P_{cr} = \frac{\pi^2 D_{ave}}{L^2}. \tag{21}$$

For the second case (Elishakoff's shape) we have

$$D_{ave} = \frac{7}{6}d_0. \tag{22}$$

The buckling load is from Eq. (15)

$$P_{cr} = \frac{72D_{ave}}{7L^2} \tag{23}$$

which is 4.2% higher than the uniform column.

For the third case (the first new solution) we have

$$D_{ave} = \frac{3}{2}d_0. \tag{24}$$

The buckling load is from Eq. (16)

$$P_{cr} = \frac{10D_{ave}}{L^2} \tag{25}$$

exactly as for the Duncan case.

For the fourth case (second new solution) we have again

$$D_{ave} = \frac{3}{2}d_0. \tag{26}$$

The buckling load is from Eq. (17)

$$P_{cr} = \frac{40D_{ave}}{L^2} \tag{27}$$

which is the same as the Duncan solution but this time for the second mode. Summarizing these results we see that the Elishakoff mode is the best (by a very slight margin).