Available online at www.sciencedirect.com
ScienceDirect
Nuclear Energy and Technology 1 (2015) 159–164
www.elsevier.com/locate/nucet

Stochastic theory of zero-power nuclear reactors. Part 3. Stochastic differential equations of zero-dimensional reactor kinetics. Weak external neutron source. Analysis of an equivalent reactivity noise model

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Available online 19 March 2016

Abstract

Stochastic differential equations of zero-dimensional reactor kinetics have been derived assuming that the discrete branching process and its continuous analog have moments of distribution converging up to the second order inclusively. The notion of weak external neutron source has been defined more accurately. Cohn's model has been comparatively analyzed against the reactivity noise model introduced by analogy with the Schottky effect.

Keywords: Stochastic differential equations; Moments of distribution; Model; Zero-power nuclear reactor; Characteristic function; Distribution; White noise.

Derivation of stochastic differential equations of zero-dimensional reactor kinetics

In [1], Eq. (8′) was obtained for the characteristic function $F(t, z_1, z_2)$ of the distribution $P_{\alpha_1,\alpha_2}(t)$ for the number α_1 of the T_1-type particles and for the number α_2 of the T_2-type particles at the time t, if these two types did not exist at $t=0$. It is obvious that, apart from the exact solution F of Eq. (8′), it is also possible to obtain the approximate value $\tilde{F}$- by representing the coefficients at F, $\frac{\partial F}{\partial z_1}$, $\frac{\partial F}{\partial z_2}$ in the equation by respective approximate expressions. For example, it is possible to expand these coefficients in a Taylor series in the powers of z_1 и z_2 and, while leaving the needed number of summands in these series (say, n), to obtain approximate values $\tilde{F}_n$ for $F(t, z_1, z_2)$. The index n, as we shall see at a later stage, does not only denote the degree of approximation, but also defines the maximal order for the coincident initial moments of the distribution $P_{\alpha_1,\alpha_2}(t)$ which are obtained from the exact solution F and the approximate solution $\tilde{F}_n$ of Eq. (8′).

One can easily make sure from Eq. (8′) that, e.g. for $\tilde{F}_2(t, z_1, z_2)$, Eq. (1) is true

$$\frac{\partial \tilde{F}_2}{\partial t} + \frac{\partial \tilde{F}_2}{\partial z_1}\Bigg\{ jz_1z_2\frac{k\beta}{L}\left[1 - \frac{\overline{\nu(\nu-1)}}{\overline{\nu}}(1-\beta)\right] - z_1\frac{k(1-\beta)-1}{L} - z_2\frac{k\beta}{L} - \frac{jz_1{}^2}{2L} \times \left[1 + k(1-\beta)\left(\frac{\overline{\nu(\nu-1)}}{\overline{\nu}}(1-\beta) - 1\right)\right] - jz_2^2\frac{k\beta}{L}\left[\frac{\overline{\nu(\nu-1)}}{\overline{\nu}}\beta + 1\right]\Bigg\} + \lambda\frac{\partial \tilde{F}_2}{\partial z_2} \times \left(jz_1z_2 - z_1 + z_2 - \frac{1}{2}jz_1^2 - \frac{1}{2}jz_2^2\right) + S\tilde{F}_2\left(\frac{1}{2}z_1^2 - jz_1\right) = 0 \qquad (1)$$

which has the boundary conditions

$$\tilde{F}_2(0, z_1, z_2) = \tilde{F}_2(t, 0, 0) = 1 \qquad (2)$$

similar to conditions (9) in [1].

The obtained equation for $\tilde{F}_2(t, z_1, z_2)$ is the equation in first-order partial derivatives with coefficients containing powers of z_1 and z_2 that do not exceed 2. Therefore, according

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Peer-review under responsibility of National Research Nuclear University MEPhI (Moscow Engineering Physics Institute).

http://dx.doi.org/10.1016/j.nucet.2015.11.009
2452-3038/

to [2], this equation, along with conditions (2), is the equation for the characteristic function of the two-dimensional diffusion process $y(t) = \{y_1(t), y_2(t)\}$, the paths for which are described by stochastic differential equations (3)

$$\frac{dy_1}{dt} = \frac{k(1-\beta)-1}{L}y_1 + \lambda y_2 + S + g_{11}(y)\xi_1(t) + g_{12}(y)\xi_2(t)$$
$$\frac{dy_2}{dt} = \frac{k\beta}{L}y_1 - \lambda y_2 + g_{21}(y)\xi_1(t) + g_{22}(y)\xi_2(t) \quad (3)$$

where $\xi_1(t)$, $\xi_2(t)$ are standard Gaussian independent white noises, and g_{ij} are the solutions of the matrix equation

$$GG^T = \mathbf{B}, \quad (4)$$

where $\boldsymbol{B}$ is the matrix with the entries

$$b_{11} = S + \frac{1}{L}\left\{1 + k(1-\beta)\left[\frac{\overline{\nu(\nu-1)}}{\overline{\nu}}(1-\beta) - 1\right]\right\}y_1 + \lambda y_2,$$

$$b_{12} = b_{21} = \frac{k\beta}{L}\left[\frac{\overline{\nu(\nu-1)}}{\overline{\nu}}(1-\beta) - 1\right]y_1 - \lambda y_2,$$

$$b_{22} = \frac{k\beta}{L}\left[\frac{\overline{\nu(\nu-1)}}{\overline{\nu}}\beta + 1\right]y_1 + \lambda y_2.$$

It is possible to write a generalization for m groups of delayed neutrons for the vector continuous random process $y(t) = \{y_1(t), ..., y_{m+1}(t)\}$:

$$\frac{dy_1}{dt} = \frac{k(1-\beta)-1}{L}y_1 + \sum_{i=2}^{m+1}\lambda_i y_i + S + \sum_{i=1}^{m+1} g_{1i}(y)\xi_i(t),$$
$$\frac{dy_i}{dt} = \frac{k\beta_i}{L}y_i - \lambda_i y_i + \sum_{j=1}^{m+1} g_{ij}(y)\xi_j(t),\ i = \overline{2, m+1} \quad (3a)$$

where g_{ij} are the entries of matrix Eq. (4) with the entries of the $(m+1)^{\text{th}}$-order matrix $\boldsymbol{B}$ of the following form:

$$b_{11} = S + \frac{1}{L}\left\{1 + k(1-\beta)\left[\frac{\overline{\nu(\nu-1)}}{\overline{\nu}}(1-\beta) - 1\right]\right\}y_1 + \sum_{i=2}^{m+1}\lambda_i y_i,$$

$$b_{1p} = b_{21} = \frac{k\beta_n}{L}\left[\frac{\overline{\nu(\nu-1)}}{\overline{\nu}}(1-\beta) - 1\right]y_1 - \lambda_p y_p,$$
$$p = \overline{2, m+1},$$
$$b_{p,l} = \frac{k\beta_p\beta_l}{L}\frac{\overline{\nu(\nu-1)}}{\overline{\nu}}y_1,\ p \neq l,\ p, l = \overline{2, m+1},$$
$$b_{pp} = \frac{k\beta_p}{L}\left[\frac{\overline{\nu(\nu-1)}}{\overline{\nu}}\beta_p + 1\right]y_1 + \lambda_p y_p,\ p = \overline{2, m+1}. \quad (4)$$

We shall use a ratio for the ith-order initial moments $M^{(n)}_{i-l,l}(t)$ of the distribution defined by the standard function $\tilde{F}_n(t, z_1, z_2)$ [3]:

$$M^{(n)}_{i-l,l}(t) = (-1)^i \left.\frac{\partial^i \tilde{F}_n}{\partial z_1^{i-1} z_2^i}\right|_{Z_1=Z_2=0} \quad (5)$$

It is clear that, with $n\to\infty$, the moments $M^{(n)}_{i-l,l}(t)$ become the moments of the exact distribution defined by the characteristic function $F(t, z_1, z_2)$.

By writing from (8′) the equation for $\tilde{F}_n(t, z_1, z_2)$, differentiating it with respect to z_1, z_2 as many times as needed, and equating the coefficients to zero, with the summands having equal powers z_1 and z_2, one can obtain the system of equations for the moments $M^{(n)}_{i-l,l}(t)$:

$$\frac{dM^{(n)}_{i-l,l}}{dt} = \frac{1-\pi}{L}\sum_{i=\chi}^{j}(-1)^{j-i}C^{j-i}_{j-l}M^{(n)}_{i+1-l,l}$$
$$+ \frac{\pi}{L}\sum_{k=0}^{u}\ \sum_{p=\max(0,k-j+l)}^{\min(k,j)}\ \sum_{i=\max(j-n-k,l-p)}^{j-k}(-1)^{j-k-l}$$
$$\times C^p_l C^{i+k-l}_{j-l} C^{k-p}_{i+k-l}\gamma_{k-p,p}M^{(n)}_{i+p+1-l,l}$$
$$+ \lambda\sum_{i=0}^{u'}\ \sum_{k=\chi-l+1}^{j-l+1}(-1)^i C^i_l C^{k+i-l-1}_{j-l}M^{(n)}_{k+i-l-1,l+1-i}$$
$$+ S\sum_{i=\chi}^{f}C^{i-l}_{j-1}M^{(n)}_{i-l,l} - \frac{1}{L}M^{(n)}_{i-l,l}$$
$$- \lambda M^{(n)}_{j-l,l+1} - SM^{(n)}_{j-l,l} \quad (6)$$

where $\chi = \max(j-n, l)$, $u = \min(n, j)$, $u' = \min(n, l)$, $q_{i,j}$are the initial moments for the distribution q_{α_1,α_2} of the numbers α_1 (prompt neutrons) and α_2 (precursors of delayed neutrons generated in one fission event).

It can be seen from system of Eqs. (6) that: (1) the system is linear; (2) all moments $M^{(n)}_{gg}(t)$ in its right-hand side have an order of not higher than j, that is, it is closed; and (3) the moments $M^{(n)}_{j-l,l}(t)$ obtained from the approximate solution $\tilde{F}_2(t, z_1, z_2)$ and the moments $M_{j-l,l}(t)$ found by the exact solution $F(t, z_1, z_2)$ of Eq. (8′), are described for $j \leq n$ by one and the same system of equation. Therefore, the moments of the distribution $P_{\alpha_1,\alpha_2}(t)$ obtained from the exact solution $F(t, z_1, z_2)$ and the approximate solution $\tilde{F}_2(t, z_1, z_2)$ of Eq. (8′) agree up to the order n inclusively.

One can judge about the quality (accuracy) of the selected approximation from the number n of coincident moments. Besides, the system of equations (6) can be always used to estimate the error when finding moments of a higher order based on an approximate model. Therefore, the continuous analog $y(t) = \{y_1(t), y_2(t)\}$ of the discrete random process $\alpha(t) = \{\alpha_1(t), \alpha_2(t)\}$ obtained at n=2, has probabilistic characteristics that agree with the same characteristics of the discrete process to within the second-order moments inclusively. Therefore, the diffusion process $y(t)$, the paths for which are described by system of equations (3), has a mathematical expectation (mean) and the second moments (dispersion, correlation functions) that agree exactly with the mathematical expectation and the second moments of the discrete numbers of prompt neutrons and precursors.