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An application of a representation theorem for fuzzy metrics to domain theory

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Abstract

In this paper we study the interplay between fuzzy metric spaces and domain theory using the representation theorem established in Mardones-Pérez and de Prada Vicente (2012) [12]. We will prove that suitable restrictions of completeness in fuzzy metric spaces correspond to a structure of domain in the poset of formal balls endowed with appropriate orders.

Keywords: Pseudometric; Fuzzy (pseudo)metric space; Completeness; Formal balls; Domain; Domain environment; Domain representable; Model

0. Introduction

It is known that topology and computer science strongly influence each other. Some topological tools are useful to investigate certain aspects of computer science, and reciprocally, the problems arising in computational setting generate new topological investigations.

This remarkable fact has motivated a great amount of research on the relations between both disciplines, as for instance [25,6,23,15,11,14,9,3]. The relations between the theory of metric spaces and domain theory, which include the construction of (computational) models for metric spaces, have deserved particular interest.

In this paper we focus our attention on the connections between fuzzy metric spaces and domain theory.

Edalat and Heckmann introduced and studied in [4] the domain of formal balls $BX = X \times [0, \infty)$ to provide a model for complete metric spaces. In their paper, completeness of metric structures and directed completeness of their posets of formal balls were shown to be in a nice correspondence. They proved, among other things, that a metric space (X, d) is complete if and only if the set of the formal balls BX ordered by the relation $(x, r) \sqsubseteq_d (y, s)$ whenever $d(x, y) \leq r - s$ is a domain. The results of Edalat and Heckmann have also been generalized to other kind of metric structures, like for instance quasi-metric spaces [1,19] or partial metric spaces [24,20]. Our main goal in this paper will be to extend to fuzzy metric spaces the results obtained by Edalat and Heckmann in [4]. This study was initiated

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by Ricarte and Romaguera [18,17] for a slightly different class of fuzzy metric spaces and with different techniques to the ones we shall use herein.

In this paper, we deal with a type of fuzzy (pseudo)metric spaces which can be represented by a certain $(0, 1)$-indexed family of (pseudo)metric spaces. Inspired by the work of Edalat and Heckmann [4], we extend to a family of (pseudo)metrics the results they achieved for a single metric. This is done in such a way that with appropriate orders on the set of formal balls, certain type of completeness of the fuzzy metric space turns out to be in correspondence with the structure of domain in the set of formal balls.

The paper is organized as follows. Section 1 begins with some preliminaries, which allow us to establish the notation and recall some background on both KM-fuzzy (pseudo)metric spaces and posets. We also collect in this section all the information we shall need concerning the main tool used in the paper: the representation theorem given in [12]. Then, Section 2 is devoted to investigate three partial order relations in the set of formal balls of a KM-fuzzy metric space which are defined in terms of the representation theorem. For the first two, several properties of the poset of formal balls BX are studied and it is shown that each relation makes BX into a domain if and only if the KM-fuzzy metric space $(X, m, \wedge)$ satisfies some suitable restrictions of completeness. The last part of the section deals with the study of a third partial order on BX motivated by a distinguished metric associated to any KM-fuzzy metric space $(X, m, \wedge)$. It is finally shown that this last approach provides a model for the topological space associated to the KM-fuzzy metric space.

1. Preliminaries

1.1. KM-fuzzy pseudometric spaces

In what follows, we recall some basic notions that will be used along the paper.

A *pseudometric* on a set X is a non-negative real-valued map d on $X \times X$ satisfying for all $x, y, z \in X$: (d1) $d(x, x) = 0$; (d2) $d(x, y) = d(y, x)$; (d3) $d(x, z) \le d(x, y) + d(y, z)$. If d satisfies the additional condition (d4) $d(x, y) = 0 \Longrightarrow x = y$, then d is a metric.

A t*-norm* T is a binary operation on $[0, 1]$ satisfying the following conditions: (T1) T is associative and commutative; (T2) $T(a, 1) = a$ for every $a \in [0, 1]$; (T3) $T(c, d) \le T(a, b)$ whenever $c \le a$, $d \le b$, with $a, b, c, d \in [0, 1]$. Observe that $T(a, 0) = 0$ and $T(a, b) \le a \wedge b$ for any $a, b \in [0, 1]$.

Basic examples of t-norms which shall be used in this paper are the minimum operator $T = \wedge$, and Lukasiewicz's t-norm, T_L, defined as $T_L(a, b) = \max\{a + b - 1, 0\}$ for each $a, b \in [0, 1]$.

We shall work with a variant of the fuzzy (pseudo)metric spaces defined in [10].

Definition 1.1. A *KM-fuzzy pseudometric space* on a set X is a triple (X, m, T) where X is a nonempty set, T is a continuous t-norm and $m : X \times X \times [0, \infty) \to [0, 1]$ is a map satisfying for all $x, y, z \in X$ and $t, s \in [0, \infty)$ the following conditions:

(FM1) $m(x, y, 0) = 0$;
(FM2) $m(x, x, t) = 1$ for any $t > 0$;
(FM3) $m(x, y, t) = m(y, x, t)$;
(FM4) $T(m(x, y, t), m(y, z, s)) \le m(x, z, t + s)$;
(FM5) $m(x, y, -) : [0, \infty) \to [0, 1]$ is left-continuous, and $\lim_{t\to\infty} m(x, y, t) = 1$.

The map m will be called a *KM-fuzzy pseudometric*. If the map m also satisfies:

(FM6) $m(x, y, t) = 1, \forall t > 0 \Longrightarrow x = y$,

then (X, m, T) is said to be a KM-fuzzy metric space.

For any KM-fuzzy pseudometric space (X, m, T) the collection $\mathcal{B} = \{U(t) : t \in (0, 1]\}$, where $U(t) = \{(x, y) \in X \times X : m(x, y, t) > 1 - t\}$ is a basis for a separated uniformity, $\mathfrak{U}(m)$, on X. It is obvious that $\{U(1/n) : n \in \mathbb{N}\}$ is a countable basis for $\mathfrak{U}(m)$. We shall denote by $\mathfrak{T}(m)$ the associated topology.