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ScienceDirect
Fuzzy Sets and Systems 299 (2016) 140–145
FUZZY sets and systems
www.elsevier.com/locate/fss

Short Communication

A note on decomposition of idempotent uninorms into an ordinal sum of singleton semigroups

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Received 10 August 2015; received in revised form 9 April 2016; accepted 11 April 2016

Available online 16 April 2016

Abstract

The idempotent uninorms are characterized by means of the ordinal sum of Clifford. It is shown that idempotent uninorms are in one-to-one correspondence with special linear orders on [0, 1]. A connection between respective linear order on [0, 1] and the characterizing multi-function of the uninorm is also investigated.

Keywords: Uninorm; Idempotent element; Internal uninorm; Ordinal sum

1. Introduction, basic notions and results

The uninorms (see [10,12,13,17,22]) generalize both t-norms and t-conorms (see [1,14]). A uninorm is a binary operation $U: [0,1]^2 \longrightarrow [0,1]$ that is commutative, associative, non-decreasing in both coordinates and has a neutral element $e \in]0,1[$. Due to the associativity, n-ary form of any uninorm is uniquely given and thus it can be extended into an aggregation function working on $\bigcup_{n\in\mathbb{N}}[0,1]^n$.

If we take uninorm in a broader sense, i.e., if for a neutral element we have $e \in [0,1]$, then the class of uninorms covers also the class of t-norms (here $e = 1$) and the class of t-conorms (here $e = 0$). For each uninorm the value $U(1,0) \in \{0,1\}$ is the annihilator of U. A uninorm is said to be *conjunctive* (*disjunctive*) if $U(1,0) = 0$ ($U(1,0) = 1$).

For each uninorm U with the neutral element $e \in]0,1[$ the restriction of U to $[0,e]^2$ is a t-norm on $[0,e]^2$, i.e., a linear transformation of some t-norm T_U on $[0,1]^2$ and the restriction of U to $[e,1]^2$ is a t-conorm on $[e,1]^2$, i.e., a linear transformation of some t-conorm S_U. Moreover, $\min(x,y) \le U(x,y) \le \max(x,y)$ for all $(x,y) \in [0,e] \times [e,1] \cup [e,1] \times [0,e]$.

On the other hand, from any pair of a t-norm and a t-conorm we can construct the minimal and the maximal uninorm with the given underlying functions.

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http://dx.doi.org/10.1016/j.fss.2016.04.007
0165-0114/

Proposition 1. *(See [15].) Let* $T\colon [0,1]^2 \longrightarrow [0,1]$ *be a t-norm and* $S\colon [0,1]^2 \longrightarrow [0,1]$ *a t-conorm and assume* $e \in [0,1]$. *Then the two functions* $U_{\min}, U_{\max}\colon [0,1]^2 \longrightarrow [0,1]$ *given by*

$$U_{\min}(x,y) = \begin{cases} e \cdot T(\frac{x}{e}, \frac{y}{e}) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e) \cdot S(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x,y) \in [e,1]^2, \\ \min(x,y) & \text{otherwise} \end{cases}$$

and

$$U_{\max}(x,y) = \begin{cases} e \cdot T(\frac{x}{e}, \frac{y}{e}) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e) \cdot S(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x,y) \in [e,1]^2, \\ \max(x,y) & \text{otherwise} \end{cases}$$

are uninorms. We will denote the set of all uninorms of the first type by $\mathcal{U}_{\min}$ *and of the second type by* $\mathcal{U}_{\max}$.

One important subclass of uninorms are idempotent uninorms, i.e., uninorms where $U(x,x) = x$ for all $x \in [0,1]$. In the case of t-norms and t-conorms there is only one idempotent t-norm – the minimum, and only one idempotent t-conorm – the maximum. Therefore idempotent uninorms are uniquely given and continuous on $[0,e]^2 \cup [e,1]^2$. Idempotent uninorms were studied in several papers (see Refs., [7,9,16,21] and references therein).

Lemma 1. *(See [9].) Let* $U\colon [0,1]^2 \longrightarrow [0,1]$ *be an idempotent uninorm. Then* U *is internal, i.e.,* $U(x,y) \in \{x,y\}$ *holds for all* $(x,y) \in [0,1]^2$.

Further, idempotent uninorms that are left-continuous, or right-continuous were characterized in [7]. Idempotent uninorms on finite ordinal scales were studied in [5]. The complete characterization of idempotent uninorms from [16] was later corrected in [21]. In the following a non-increasing function $g\colon [0,1] \longrightarrow [0,1]$ is called Id-symmetrical if its completed graph F_g is Id-symmetrical, i.e., $(x,y) \in F_g$ if and only if $(y,x) \in F_g$. Note that a completed graph was defined in [21] as follows: let $g\colon [0,1] \longrightarrow [0,1]$ be any decreasing function and let G be the graph of g, that is

$$G = \{(x, g(x)) \mid x \in [0,1]\};$$

for any point of discontinuity s of g, let s^- and s^+ be the corresponding lateral limits. Then, we define the completed graph of g, denoted by F_g, as the set obtained from G by adding the vertical segments in any discontinuity point s, from s^- to s^+.

Theorem 1. *(See [21].) Consider* $e \in \,]0,1[$. *The following items are equivalent:*

(i) U *is an idempotent uninorm with neutral element* e.
(ii) There exists a decreasing, Id-symmetrical function $g\colon [0,1] \longrightarrow [0,1]$ *with fixed point* e *such that* U *is for all* $(x,y) \in [0,1]^2$ *given by*

$$U(x,y) = \begin{cases} \min(x,y) & \text{if } y < g(x) \text{ or } y = g(x), x < g(g(x)), \\ \max(x,y) & \text{if } y > g(x) \text{ or } y = g(x), x > g(g(x)), \\ x \text{ or } y & \text{if } y = g(x), x = g(g(x)), \end{cases}$$

being commutative on the set of points $(x, g(x))$ *such that* $x = g(g(x))$.

Note that the function g coincides with the characterizing multi-function of U which we now recall.

Definition 1. (See [18].) A mapping $p\colon [0,1] \longrightarrow \mathcal{P}([0,1])$ is called a multi-function if to every $x \in [0,1]$ it assigns a subset of $[0,1]$, i.e., $p(x) \subseteq [0,1]$. A multi-function p is called