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Fuzzy Sets and Systems 295 (2016) 114–135

FUZZY sets and systems

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On the solutions to first order linear fuzzy differential equations

A. Khastan [a,*], R. Rodríguez-López [b]

[a] Department of Mathematics, Institute for Advanced Studies in Basic Sciences (IASBS), Zanjan 45137-66731, Iran

[b] Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Santiago de Compostela, 15782 Santiago de Compostela, Spain

Received 16 February 2014; received in revised form 17 April 2015; accepted 11 June 2015

Available online 16 June 2015

Abstract

In this paper, we study different formulations of first order linear fuzzy differential equations using the concept of generalized differentiability. We present sufficient conditions for the existence of solutions and obtain the general expression of these solutions, which exhibit different behavior. Some examples are given to illustrate our results.

Keywords: First order fuzzy differential equations; Generalized differentiability; Fuzzy linear differential equations

1. Introduction

It is proposed by Bhaskar et al. [2] that the introduction of an appropriate forcing term in a fuzzy differential equation can be used to control the behavior of the solution. Under the traditional approach in the interpretation of fuzzy differential equations, their point of view is that, when some specific kind of phenomena are incorporated into an ordinary differential equation, we should not expect the same behavior for the solution to the new system in comparison with the old (classical) one. Hence, they defend that research in fuzzy differential equations is interesting itself as a proper and independent discipline. They illustrate their approach by starting from a simple initial value problem for ordinary differential equations given in the following two equivalent forms:

$$x'(t) = -x(t),\ x(0) = x_0 \quad \text{and} \quad x'(t) + x(t) = \tilde{0},\ x(0) = x_0,$$

which, however, in the fuzzy setting, are no more equivalent. To fuzzify the second equation avoiding trivial cases, the authors in [2] introduce a forcing term σ into the equation, obtaining the fuzzy differential equation $x'(t) + x(t) = \sigma(t)$, $x(0) = x_0$, $t \geq 0$.

The work of Bhaskar et al. [2] was later extended by Kaleva [11] and Bede et al. [3], using different interpretations. Some other related references which deal with various formulations and interpretations of fuzzy initial value problems are, for instance, [1,5,7,14,16,17,19,20].

* Corresponding author.
E-mail addresses: khastan@iasbs.ac.ir, khastan@gmail.com (A. Khastan), rosana.rodriguez.lopez@usc.es (R. Rodríguez-López).

http://dx.doi.org/10.1016/j.fss.2015.06.005

In [5], the same modelling problem was raised by considering the fuzzy initial value problem

$$\begin{cases} y'(t) = a(t)y(t) + b(t), & t \in I, \\ y(0) = y_0, \end{cases} \tag{1}$$

where $y_0 \in \mathbb{R}_F$, $a : I \to \mathbb{R}$ and $b : I \to \mathbb{R}_F$. It is well-known that this problem is not equivalent in general to any of the following other two problems:

$$\begin{cases} y'(t) + (-a(t))y(t) = b(t), & t \in I, \\ y(0) = y_0 \end{cases} \tag{2}$$

and

$$\begin{cases} y'(t) + (-b(t)) = a(t)y(t), & t \in I, \\ y(0) = y_0. \end{cases} \tag{3}$$

In the general case, these problems (2) and (3) are neither equivalent. Throughout the paper, when referring to problems (1)–(3), we consider $y_0 \in \mathbb{R}_F$ and $a : I \to \mathbb{R}$, $b : I \to \mathbb{R}_F$ continuous functions on I (I an interval of the type $(0, T)$, although $(0, +\infty)$ is not excluded).

At this point, a question which deserves attention is, as remarked in [8] for linear interval differential equations, trying to discern, among the different above-mentioned problems, the formulation which can be denominated the natural fuzzy correspondent to the initial value problem for the crisp linear differential equation of origin. In relation with this, we recall that a usual procedure to introduce uncertainty in a dynamical system in order to predict the behavior of imprecise real-world phenomena is the fuzzification of the corresponding crisp differential equations. This leads to the situation where, depending on the different formulations of the unique crisp equation, we may get three different results through fuzzification. This situation is unnatural and in some sense contradictory since, in any model, we expect that the solution reflects faithfully the real behavior of the system, independently of the similar particular formulations of the equation. However, this fact is found an advantage by the authors of [2], due to the existence of several choices which can be examined through the scrutiny of the physical features of the particular phenomena. Here, we do not give an answer to the question about which is the best formulation for the problem or the closest to the classical case, in fact, we restrict our study to the analysis of the different inequivalent fuzzy versions of the crisp problem (1), (2) and (3), providing sufficient conditions for the existence of solution and, in the positive case, the explicit expressions of their solutions.

Therefore, in this paper, we consider these three different inequivalent formulations of first order linear fuzzy differential equations under the generalized differentiability concept. Our results complement some included in [3,5], where the initial value problems related to the fuzzy differential equations (1), (2) and (3) were studied. In this work, we complete the analysis, showing new expressions for solutions which are not included in [5]. We agree with the hypothesis in [3], where it is explained that, depending on the properties of the particular phenomena of study, one can try to select an appropriate formulation of the problem, trying to adjust better the response of the fuzzy model. This way, the solutions obtained can be used as a mechanism to compare their properties and to choose what can be considered the most adequate formulation to model the behavior of a particular real phenomena through a dynamical system with uncertainty. Moreover, as we show in the main section, the same fuzzy function can be a candidate for a solution to two of the problems considered depending on the type of (generalized) differentiability satisfied, which illustrates some connections between these a priori inequivalent linear problems, providing also some clues in the analysis of the variety and richness of the theory of fuzzy differential equations. Several examples are presented to illustrate the applicability of our results.

2. Preliminaries

In this section, we give some definitions and useful results and introduce the necessary notation which will be used throughout the paper. Most of it can be found, for example, in [4,9,10].

In the following, we denote the space of fuzzy intervals by $\mathbb{R}_F$. Given a fuzzy interval $u \in \mathbb{R}_F$ and $0 < \alpha \leq 1$, we obtain the α-level set of u by $[u]^\alpha = \{s \in \mathbb{R} \mid u(s) \geq \alpha\}$ and the support of u as $[u]^0 = cl\{s \in \mathbb{R} \mid u(s) > 0\}$. For any $\alpha \in [0, 1]$, due to the properties imposed on the set of fuzzy intervals, we have that $[u]^\alpha$ is a bounded closed interval. The notation $[u]^\alpha = [\underline{u}^\alpha, \overline{u}^\alpha]$, denotes explicitly the α-level set of u. For $u \in \mathbb{R}_F$, we define the length of u as: $diam(u) = \overline{u}^\alpha - \underline{u}^\alpha$.