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Cores and Weber sets for fuzzy extensions of cooperative games ☆

Nobusumi Sagara [1]

Faculty of Economics, Hosei University, 4342, Aihara, Machida, Tokyo, 194-0298, Japan

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Abstract

This paper investigates fuzzy extensions of cooperative games and the coincidence of the solutions for fuzzy and crisp games. We show that an exact game has an exact fuzzy extension such that its fuzzy core coincides with the core. For games with empty cores, we exploit Lovász extensions to establish the coincidence of Weber sets for fuzzy and crisp games.

Keywords: Core; Weber set; Exact game; Superdifferential; Lovász extension; Supermodularity; Convex games

1. Introduction

Classical cooperative game theory deals with the situation where players have only two alternative possibilities. That is, whether they join a coalition or not, without any option for the degree of commitment. Further, once the players join a coalition, all of them are required to commit themselves fully to the coalition. On the contrary, cooperative fuzzy games proposed by [1,2] allow for the partial participation of players in coalitions where the attainable outcomes of a game depend on the degree of commitment of the players, thereby modeling ambiguous decision making as observed in most group behavior.

Fuzzy games are defined on fuzzy coalitions. The restriction of fuzzy games to usual coalitions (characteristic functions) yields a crisp (or nonfuzzy) game. However, there are numerous fuzzy games that are fuzzy extensions of a given game, and there are innumerable fuzzy games that yield the same crisp game. Similarly to classical cooperative game theory, the fuzzy core is a fundamental solution concept in fuzzy game theory. Since the core of a crisp game

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E-mail address: nsagara@hosei.ac.jp.

[1] Tel.: +81 42 7832556; fax: +81 42 7832611.

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is sometimes too large in applications, the fact that the fuzzy core of a fuzzy extension is included in the core of its original (or crisp) game seems to be a desirable property. In particular, it is well-known that in exchange economies the set of core allocations is larger than that of Walrasian allocations, but the set of fuzzy core allocations coincides with that of Walrasian allocations (see [1,2,11]).

Notwithstanding this observation, we stress the point that the "shrinkage" of the core depends crucially upon the choice of fuzzy extensions of a crisp game, and it never arises from some typical fuzzy extensions. The first question addressed in this paper is as follows. Under what conditions in crisp games and their fuzzy extensions is the coincidence of the core of a fuzzy game and that of a crisp game guaranteed?

The response given in the paper is that a consistent class of games and their fuzzy extensions that guarantee the core coincidence is the class of exact games and their exact fuzzy extensions investigated in [15,22,23]. For the core coincidence, the useful observation made by [1] that the superdifferential of a positively homogeneous fuzzy game is the fuzzy core plays a crucial role because the above extensions are positively homogeneous, concave functions and the standard convex analysis are effectively used. Exact fuzzy extensions are very different from multilinear extensions proposed by [20] in that the former are superlinear and not necessarily differentiable, but the latter are continuously differentiable and not positively homogeneous.

If the core of a crisp game is empty, which can happen in many applications, then the core coincidence is utterly in vain. In this case, what kind of solutions and fuzzy extensions should be pursued for the coincidence of the solutions for fuzzy and crisp games? This is the second question addressed in this paper. To answer this natural question, we propose to exploit Weber sets and Lovász extensions (or Choquet integrals) for the class of "all" crisp games. The advantage of Weber sets is exemplified by the fact that the Weber set of any crisp game is nonempty and contains the core and the Shapley value (see [28]).

The Weber set is defined as the convex hull of the marginal contributions of each player. We define the fuzzy Weber set of a fuzzy game via the limiting marginal contributions of each player in terms of the directional derivative of a fuzzy game and demonstrate that the fuzzy Weber set of the Lovász extension of any crisp game coincides with the Weber set. Toward this end, we prove that the Clarke superdifferential of the Lovász extension is the fuzzy Weber set. The fact that Lovász extensions are not necessarily concave or differentiable furnishes one with another reason to utilize the Clarke superdifferential, a generalized notion of the superdifferentials in convex analysis (see [7]), to investigate Weber sets. Based on the useful technique of nonsmooth analysis brought into cooperative game theory in [3], we provide an additional characterization of convex games in terms of the regularity of the Lovász extensions, as a weaker form of smoothness.

The organization of the paper is as follows: Section 2 collects preliminary notions and results on nonsmooth analysis employed throughout the paper and summarize the well-known results on exact games, cores, Weber sets, and fuzzy games. The first question on the core coincidence is answered in Section 3. The main part of the paper is in Section 4, where the second question on the coincidence of Weber sets is systematically explored. Appendix A provides the proofs of the two key lemmas employed in Section 4.

2. Preliminaries

2.1. Clarke superdifferentials

The *directional derivative* of a real-valued function $f:\mathbb{R}^n\to\mathbb{R}$ at $x\in\mathbb{R}^n$ in the direction $h\in\mathbb{R}^n$ is defined by

$$f'(x;h)=\lim_{\lambda\downarrow 0}\frac{f(x+\lambda h)-f(x)}{\lambda}$$

when this limit exists in $\mathbb{R}$. The *superdifferential* $\partial f(x)$ of f at x is given by

$$\partial f(x)=\left\{p\in\mathbb{R}^n \;\middle|\; f(y)-f(x)\le\langle p,y-x\rangle\ \forall y\in\mathbb{R}^n\right\},$$

where we denote by $\langle x,y\rangle$ the inner product of the vectors $x,y\in\mathbb{R}^n$. An element in $\partial f(x)$ is called a *supergradient* of f at x. It follows from the standard argument of convex analysis in finite dimensions that if f is concave and $f(x)$