Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Sets and Systems 272 (2015) 115–125

FUZZY sets and systems

www.elsevier.com/locate/fss

Games with fuzzy authorization structure: A Shapley value

J.M. Gallardo, N. Jiménez, A. Jiménez-Losada *, E. Lebrón

Escuela Técnica Superior de Ingeniería, Matemática Aplicada II, Universidad de Sevilla, Camino de los Descubrimientos, 41092 Sevilla, Spain

Received 23 October 2013; received in revised form 22 May 2014; accepted 1 September 2014

Available online 6 September 2014

Abstract

A cooperative game consists of a set of players and a characteristic function which determines the maximal gain or minimal cost that every subset of players can achieve when they decide to cooperate, regardless of the actions that the other players take. It is often assumed that the players are free to participate in any coalition, but in some situations there are dependency relationships among the players that restrict their capacity to cooperate within some coalitions. Those relationships must be taken into account if we want to distribute the profits fairly. In this respect, several models have been proposed in literature. In all of them dependency relationships are considered to be complete, in the sense that either a player is allowed to fully cooperate within a coalition or they cannot cooperate at all. Nevertheless, in some situations it is possible to consider another option: that a player has a degree of freedom to cooperate within a coalition. A model for those situations is presented.

Keywords: Cooperative game; Fuzzy coalition; Shapley value; Fuzzy authorization structure

1. Introduction

In a general way, game theory studies cooperation and conflict models, using mathematical methods. This paper is about cooperative game theory. A cooperative game over a finite set of players is defined as a function establishing the worth of each coalition. Given a cooperative game, the main problem that arises is how to assign a payoff to each player in a reasonable way. In this setting, it is often assumed that all of the players are socially identical. In real life, however, political or economic circumstances may impose certain restraints on coalition formation. This idea has led several authors to develop models of games in which relationships among players must be taken into account. Depending on the nature of such relationships, different structures in the set of players have been considered. Myerson [12] studied games in which communication between players is restricted. He considered graphs to model those restraints. Subsequently, different kinds of limitations on cooperation among players have been studied, and various structures have been used for that, like convex geometries (see [3]), matroids (see [4]), antimatroids (see [1]) or augmenting systems (see [5]). A particularly interesting case of limited cooperation arises when we consider veto relationships between players. In this regard, Gilles et al. [11] modeled situations in which a hierarchical structure

* Corresponding author.
E-mail address: ajlosada@us.es (A. Jiménez-Losada).

http://dx.doi.org/10.1016/j.fss.2014.09.002

imposes some constraints on the behavior of the players in the game. They introduce games with permission structure, that consist of a set of players, a cooperative game and a mapping that assigns to every player a subset of direct subordinates. In this respect, the power of a player over a subordinate can be of different kinds. In the conjunctive approach it is assumed that each player needs the permission of all his superiors, whereas in the disjunctive approach of van den Brink [7], the permission of any of those superiors will suffice. In each case they consider a new characteristic function, which collects the information given by both the original characteristic function and the permission structure, and which allows them to define a value for games on conjunctive (or respectively disjunctive) permission structures. They provide intuitive characterizations for each case, showing in this way that the values obtained are reasonable. Subsequently, Derks and Peters [10] generalized those approaches by considering the so-called restrictions. Although their model is more general, the axiomatization given is not as intuitive and straightforward as those given by Gilles et al. [11] and van den Brink [7] for permission structures.

In all of the models presented so far the dependency relationships are complete, in the sense that either a coalition can veto a player or it does not have any authority over the player. Our aim in this paper will be to provide a new model for games in which players are subject to certain restraints when cooperating within a coalition. We will consider the possibility that such restraints are partial, which will make this model more general than those referenced above.

The paper is organized as follows. In Section 2 we recall some basic definitions and properties about the Shapley value, fuzzy sets and the Choquet integral. In Section 3, we introduce fuzzy authorization structures, that will be used to model situations in which some players depend partially on other players. Then, for each game with fuzzy authorization structure, a new characteristic function, that collects the information from both the game and the structure, is be defined. This characteristic function will allow us to define a Shapley value for games with fuzzy authorization structure. A characterization of this value is given in Section 4. An example is described as well. Finally, in Section 5 some conclusions are given.

2. Preliminaries

2.1. Cooperative TU-games

We recall some concepts regarding cooperative games. A *transferable utility cooperative game* or *TU-game* is a pair (N, v) where N is a finite set and $v : 2^N \to \mathbb{R}$ is a function with $v(\emptyset) = 0$. The elements of $N = \{1, ..., n\}$ are called players, and the subsets of N coalitions. Given a coalition E, $v(E)$ is the worth of E, and it is interpreted as the maximal gain or minimal cost that the players in this coalition can achieve by themselves against the best offensive threat by the complementary coalition. Frequently, a TU-game (N, v) is identified with the function v. A game v is monotone if for every $F \subseteq E \subseteq N$, it holds that $v(F) \leq v(E)$. The family of games with set of players N is denoted by $\mathcal{G}^N$. This set is a $(2^n - 1)$-dimensional real vector space. One basis of this space is the collection $\{u_F : F \subseteq N, F \neq \emptyset\}$ where for a nonempty coalition F the unanimity game u_F is defined by

$$u_F(E) = \begin{cases} 1 & \text{if} F \subseteq E, \\ 0 & \text{otherwise.} \end{cases}$$

Every game $v \in \mathcal{G}^N$ can be written as a linear combination of them,

$$v = \sum_{\{E \in 2^N : E \neq \emptyset\}} \triangle_v(E) u_E$$

where $\triangle_v(E)$ is the dividend of the coalition E in the game v.

A solution or value on $\mathcal{G}^N$ is a function $\psi : \mathcal{G}^N \to \mathbb{R}^N$ that assigns to each game a vector $(\psi_1(v), \ldots, \psi_n(v))$ where the real number $\psi_i(v)$ is the payoff of the player i in the game (N, v).

Many values have been defined in literature for different families of games. The *Shapley value* (see [13]) $\phi(v) \in \mathbb{R}^N$ of a game $v \in \mathcal{G}^N$ is a weighted average of the marginal contributions of each player to the coalitions and formally it is defined by

$$\phi_i(v) = \sum_{\{E \subseteq N : i \in E\}} p_E\big(v(E) - v\big(E \setminus \{i\}\big)\big), \quad \text{for all } i \in N,$$

where