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ScienceDirect
Fuzzy Sets and Systems 259 (2015) 68–88
www.elsevier.com/locate/fss

Lattice-valued bornological systems [☆]

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Received 10 September 2013; received in revised form 27 March 2014; accepted 8 September 2014

Available online 16 September 2014

Abstract

Motivated by the concept of lattice-valued topological system of J.T. Denniston, A. Melton, and S.E. Rodabaugh, which extends lattice-valued topological spaces, this paper introduces the notion of lattice-valued bornological system as a generalization of lattice-valued bornological spaces of M. Abel and A. Šostak. We aim at (and make the first steps towards) the theory, which will provide a common setting for both lattice-valued point-set and point-free bornology. In particular, we show the algebraic structure of the latter.

Keywords: Adjoint functor; (Lattice-valued) bornological space; (Lattice-valued) topological system; Locale; Localification and spatialization of topological systems; Point-free topology; Reflective subcategory

1. Introduction

In 1989, S. Vickers [34] introduced the notion of topological system as a common setting for both topological spaces, and their underlying algebraic structures – locales. Recently, J.T. Denniston, A. Melton, and S.E. Rodabaugh [6] provided the concept of lattice-valued topological system as an extension of lattice-valued topological spaces. Lattice-valued analogues of the main system-related procedures, i.e., spatialization (a space from a system) and localification (a locale from a system) were soon considered in [28,29,31], thereby providing a complete fuzzification of the original setting of S. Vickers. At present, the theory of lattice-valued topological systems has already found applications in several scientific fields [7,8,11,12,30,32], including, e.g., formal concept analysis, topology, theoretical computer science, and theoretical physics.

☆ This research was supported by the ESF Project No. CZ.1.07/2.3.00/20.0051 “Algebraic methods in Quantum Logic” of the Masaryk University in Brno, Czech Republic; and also by the Aktion Project No. 67p5 (Austria–Czech Republic) “Algebraic, fuzzy and logical aspects of statistical learning for cancer risk assessment”. The third author acknowledges support of WTZ Project No. IN 11/2011 and Aktion Project No. 63p22.
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http://dx.doi.org/10.1016/j.fss.2014.09.006

In 2011, M. Abel and A. Šostak [1] came out with a fuzzification of a well-known concept of functional analysis, namely, bornological space [14], and considered the category of such new structures. The main meta-mathematical difference between topological and bornological spaces is that the former provide a convenient tool to study “continuity”, and the latter do the same job for “boundedness”. As a result, while both notions are defined through a collection of subsets of a set, the respective axioms are quite different.

Motivated by the theory of topological systems, this paper introduces the notion of bornological system and shows its possible fuzzification. In particular, we provide bornological (and, partially, lattice-valued) analogues of system spatialization and localification procedures. It should be emphasized, however, immediately that the latter procedure requires the concept of point-free bornology. Despite the fact that the theory of point-free topology is already well-developed [17], point-free bornology (up to our knowledge) is still non-existent. To fill the gap, we introduce a possible approach to point-free bornology, e.g., describe the algebraic structure, which underlies bornologies, and show its respective homomorphism. Similar to the case of topological systems, we aim at providing a common setting for both point-set and point-free bornologies. Additionally (and as a kind of byproduct), we obtain the necessary and sufficient condition for the construct of lattice-valued fixed-basis [15] bornological spaces to be topological (correcting thereby the result of [1, Theorem 5.3], which states that given an infinitely distributive lattice L, the construct L-**Born** of L-bornological spaces and L-bounded maps is topological), as well as extend this necessary and sufficient condition to the variable-basis approach to bornological spaces (in the sense of S.E. Rodabaugh [23]).

We believe that the developed theory of bornological systems will find its applications in cancer research. To be more precise, tumors in humans can be conveniently modeled by fractals [13,19,20,33], the “dimension” of which is non-integer [4]. One of the ways how to get these non-integers in metric spaces is to use the so-called Hausdorff dimension [9], whose particular values could tell the level of carcinogenicity of a given tumor. In practical applications, however, one often encounters a bornological space instead of a metric one. Motivated by the challenge, J. Almeida and L. Barreira [3] introduced the concept of Hausdorff dimension for convex bornological spaces. Encapsulating both geometric and algebraic information, bornological systems though seem to us to be more suitable in certain cases, which will be the subject of our forthcoming papers. For convenience of the reader, however, we provide in Appendix A of the present paper a particular example of our intended developments in cancer research, which is related to (lattice-valued) bornological spaces.

2. Bornological systems

This section introduces bornological systems, following the pattern of topological systems of S. Vickers [34], and shows bornological analogues of the system spatialization and localification procedures.

2.1. Bornological spaces

Given a map $X \xrightarrow{f} Y$, we often use the *forward powerset (image) operator* $\mathcal{P}(X) \xrightarrow{f^{\rightarrow}} \mathcal{P}(Y)$, $f^{\rightarrow}(S) = \{f(s) \mid s \in S\}$, and the *backward powerset (preimage) operator* $\mathcal{P}(Y) \xrightarrow{f^{\leftarrow}} \mathcal{P}(X)$, $f^{\leftarrow}(T) = \{x \in X \mid f(x) \in T\}$ [24]. With this preliminary in hand, we recall the concept of bornological space [14].

Definition 1. A *bornological space* is a pair $(X, \mathcal{B})$, where X is a set and $\mathcal{B}$ (a *bornology* on X) is a subfamily of $\mathcal{P}(X)$, the elements of which are called *bounded sets*, and they satisfy the following axioms:

(1) $X = \bigcup \mathcal{B}$ $(= \bigcup_{B \in \mathcal{B}} B)$;
(2) if $B \in \mathcal{B}$ and $D \subseteq B$, then $D \in \mathcal{B}$;
(3) if $\mathcal{S} \subseteq \mathcal{B}$ is finite, then $\bigcup \mathcal{S} \in \mathcal{B}$.

Given bornological spaces $(X_1, \mathcal{B}_1)$ and $(X_2, \mathcal{B}_2)$, a map $X_1 \xrightarrow{f} X_2$ is called *bounded* provided that $f^{\rightarrow}(B_1) \in \mathcal{B}_2$ for every $B_1 \in \mathcal{B}_1$. **Born** is the category of bornological spaces and bounded maps, which is concrete over the category **Set** of sets and maps (i.e., is a *construct*). ■

Motivated by the theory of topological spaces and systems, we make three important remarks.