Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Sets and Systems 289 (2016) 151–156

FUZZY sets and systems

www.elsevier.com/locate/fss

Distributivity and associativity in effect algebras

Josef Tkadlec

Department of Mathematics, Faculty of Electrical Engineering, Czech Technical University in Prague, 166 27 Praha, Czech Republic

Received 17 July 2014; received in revised form 9 June 2015; accepted 22 June 2015

Available online 15 July 2015

Abstract

We prove "large associativity" of the partial sum in effect algebras and present an overview of distributivity-like properties of partial operations $\oplus$ and $\ominus$ in effect algebras with respect to (possibly infinite) suprema and infima and vice versa generalizing several previous results.

Keywords: Effect algebra; Distributivity; Associativity

1. Introduction

Effect algebras [4] and equivalent D-posets [6] were introduced in the nineties of the twentieth century as "unsharp" generalizations of "sharp" quantum logics (orthomodular lattices, orthomodular posets, orthoalgebras) incorporating some fuzzy logics (MV-algebras). E.g., consider the effect algebra $([0, 1], \oplus, 0, 1)$ with the real unit interval $[0, 1]$ and the partial operation $\oplus$ defined as the sum of real numbers whenever this sum belongs to $[0, 1]$. This effect algebra corresponds to MV-algebra with the Łukasiewicz t-conorm $\oplus$ if we extend the definition of $\oplus$ by $a \oplus b = 1$ whenever $a + b > 1$.

Effect algebras are partially ordered by a natural way. The distributivity-like properties of suprema and infima (possibly infinite) with respect to partial operations $\oplus$ and $\ominus$ and vice versa were studied by Bennett and Foulis [1] in the context of effect algebras (sometime assuming that they form a lattice) and by Chovanec and Kôpka [2] in the context of D-posets (for two-element sets assuming that the D-posets form a lattice). We present a unified overview of generalizations of these results.

A "large associativity" (also for infinite number of elements) of the partial operation $\oplus$ was studied by Riečanová [7] in the context of abelian RI-posets for complete lattices and by Ji [5] for orthocomplete effect algebras. We generalize these results for effect algebras.

We present examples showing that these results cannot be improved to obtain distributivity (associativity, resp.) in all cases.

E-mail address: tkadlec@fel.cvut.cz.

http://dx.doi.org/10.1016/j.fss.2015.06.025

Our results can be useful in the study of effect algebras (quantum and fuzzy structures)—see, e.g., [1,3,5]. They seem to be ultimate because we were able to omit all assumptions for the underlying structure and for the cardinalities of considered sets.

2. Basic notions and properties

Let us start with a review of basic notions and properties.

Definition 2.1. An *effect algebra* is an algebraic structure $(E, \oplus, \mathbf{0}, \mathbf{1})$ such that E is a set, $\mathbf{0}$ and $\mathbf{1}$ are different elements of E, and $\oplus$ is a partial binary operation on E such that for every $a, b, c \in E$ the following conditions hold:

(1) $a \oplus b = b \oplus a$, if one side exists;
(2) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$, if one side exists;
(3) there is a unique *orthosupplement* a' such that $a \oplus a' = \mathbf{1}$;
(4) $a = \mathbf{0}$ whenever $a \oplus \mathbf{1}$ is defined.

For simplicity, we will use the notation E for an effect algebra. A partial ordering on an effect algebra E is defined by $a \leq b$ if there is a $c \in E$ such that $b = a \oplus c$. Such an element c is unique (if it exists), is equal to $(a \oplus b')'$ and is denoted by $b \ominus a$. In particular, $a' = 1 \ominus a$. With respect to this partial ordering, $\mathbf{0}$ ($\mathbf{1}$, resp.) is the least (the greatest, resp.) element of E. The orthosupplementation is an antitone involution, i.e., for every $a, b \in E$, $a'' = a$ and $b' \leq a'$ whenever $a \leq b$. An *orthogonality* relation on E is defined by $a \perp b$ if $a \oplus b$ exists (that is if and only if $a \leq b'$). It can be shown that $a \oplus \mathbf{0} = a$ for every $a \in E$ and that the *cancellation law* is valid: if $a \oplus c \leq b \oplus c$ then $a \leq b$ (in particular, if $a \oplus c = b \oplus c$ then $a = b$). See, e.g., [3,4].

An equivalent notion (in the sense of a natural correspondence) of a *D-poset* defined by the properties of the partial operation $\ominus$ is used sometimes. See, e.g., [3,6].

Definition 2.2. Let E be an effect algebra. A system $(a_i)_{i \in I}$ of elements of E is *orthogonal* if $\bigoplus_{i \in F} a_i$ is defined for every finite set $F \subseteq I$. A *majorant* of an orthogonal system is an upper bound of all its finite sums. The *sum* $\bigoplus_{i \in I} a_i$ of an orthogonal system $(a_i)_{i \in I}$ is its least majorant (if it exists).

A finite system is orthogonal if and only if the sum of all its elements is defined. Every subsystem of an orthogonal system is orthogonal. The empty system is orthogonal and its sum is the least element $\mathbf{0}$. Every pair of elements in an orthogonal system is orthogonal. On the other hand there are nonorthogonal systems of pairwise orthogonal elements if (and only if) the effect algebra does not form an orthomodular poset.

A simple example of an effect algebra is the structure $([0, 1], \oplus, 0, 1)$ where $[0, 1]$ is the interval of real numbers and $\oplus$ is defined by $a \oplus b = a + b$ for $a + b \leq 1$. Then $a \ominus b = a - b$ (whenever it is defined).

Let us summarize some properties of the operations $\oplus$ and $\ominus$ showing that these partial operations behave very much like the real operations $+$ and $-$. The basic difference is that we have to take care whether they are defined.

Lemma 2.3. *Let E be an effect algebra, $a, b, c, a_i \in E$, $i \in I$, I is finite:*

(1) *If $b = \bigoplus_{i \in I} a_i$ then $b \geq \bigoplus_{i \in J} a_i$ and $b \ominus \bigoplus_{i \in J} a_i = \bigoplus_{i \in I \setminus J} a_i$ for every $J \subseteq I$. In particular, $(a \oplus b) \ominus b = a$ whenever $a \perp b$.*
(2) *If $a \leq b$ then $a \oplus (b \ominus a) = b$, $b \ominus (b \ominus a) = a$ and $b \ominus a = a' \ominus b'$.*
(3) *If $a \leq b \perp c$ then $a \oplus c \leq b \oplus c$ and $b \oplus c = (a \oplus c) \oplus (b \ominus a)$, i.e., $(b \oplus c) \ominus a = (b \ominus a) \oplus c$.*
(4) *If $a \leq b \leq c$ then $c \ominus a = (b \ominus a) \oplus (c \ominus b)$, i.e., $b \ominus a \leq c \ominus a$ and $c \ominus b \leq c \ominus a$.*
(5) *$c \ominus (a \oplus b) = (c \ominus a) \ominus b = (c \ominus b) \ominus a$ whenever one of the compared expressions exists.*
(6) *If $a \perp b$ then $(a \oplus b)' = a' \ominus b = b' \ominus a$.*

Proof. (1) It is a consequence of the commutativity and associativity of $\oplus$ and of the definition of $\ominus$.

(2) The first equality is the definition of $b \ominus a$, the second follows using part (1), the third follows from the cancellation law and from the equality (we use $b' \leq a'$) $a \oplus (b \ominus a) \oplus b' = b \oplus b' = \mathbf{1} = a \oplus a' = a \oplus b' \oplus (a' \ominus b')$.

(3) $b \oplus c = a \oplus (b \ominus a) \oplus c = (a \oplus c) \oplus (b \ominus a)$.