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# On the power sequence of a fuzzy matrix with convex combination of max-product and max-min operations

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**Abstract**

In the literature, the powers of a fuzzy matrix with max-min/max-product/max Archimedean t-norm/max t-norm/max-arithmetic mean compositions have been studied. It turns out that the limiting behavior of the powers of a fuzzy matrix depends on the composition involved. In this paper, we consider the powers of a fuzzy matrix with convex combination of max-product and max-min operations. We show that the powers of a fuzzy matrix $A$ with such operations are asymptotical $p$-period if and only if the powers of an associated Boolean matrix $\bar{A}$ are $p$-periodic. Moreover, necessary and sufficient conditions for such a fuzzy matrix to be nilpotent are proposed.


*Keywords:* Convex combination of max-product and max-min composition; Asymptotical $p$-period; Nilpotence

## 1. Introduction

Fuzzy matrices have been proposed to represent fuzzy relations on finite universes. By a fuzzy matrix $A$, we mean $A = [a_{ij}]$ with $a_{ij} \in [0, 1]$. Let $A$ be an $n \times n$ fuzzy matrix. The corresponding Boolean matrix $\bar{A}$ of $A$ is defined by

$$\bar{A}_{ij} := \begin{cases} 1 & \text{if } a_{ij} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Given $\lambda \in [0, 1]$, the convex combination of max-product and max-min operation, denoted by "$\otimes$", for matrix $A$ can be defined as

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$$[A \otimes A]_{ij} = \max_{1\le t\le n} \{a_{it} \otimes a_{tj}\} = \max_{1\le t\le n} \{\lambda a_{it}a_{tj} + (1-\lambda)\min\{a_{it}, a_{tj}\}\}, \forall 1 \le i, j \le n.$$

We note that if $\lambda = 1$ then the operation $\otimes$ becomes the commonly seen max product operation. On the other hand, if $\lambda = 0$ then $\otimes$ becomes the max-min operation. Since it is very easy to illustrate by numerical examples that the operation $\otimes$ is nonassociative, we therefore define the powers $A^k$ of $A$ by

$$A^k_\otimes = (A^{k-1}_\otimes) \otimes A,\ k = 2, 3, \cdots,$$

where $A^1_\otimes = A$.

Powers of a fuzzy matrix play a crucial role in finding the transitive closure of the underlying fuzzy relation. Thomason's paper [22] was the first to explore the powers of a fuzzy relation. He showed that the max-min powers of a fuzzy matrix [2,5–8,11–17] either converge to an idempotent matrix or oscillate with a finite period. On the other hand, the behavior of max-product powers of a fuzzy matrix is quite different from the case with the max-min fuzzy matrices [1,21]. In the literature, characterization of max-product powers of a fuzzy matrix was in terms of the notion of asymptotic period [10]. It turns out the limiting behavior of the consecutive powers can be completely determined by an associated Boolean matrix [21]. Since the powers of a Boolean matrix are easy to compute, the limiting behavior of the consecutive powers of a max-product fuzzy matrix can be determined efficiently. It is well known that the max-product operation is one of the max-Archimedean t-norms. This Boolean characterization has been extended to the environment with max-Archimedean t-norms [20].

Various publications [2,3,6–10,12] evaluated the powers of a fuzzy matrix with max-min/max-product/lattice compositions and studied the relative properties or applications with these compositions in the literatures, however the limiting behavior of the powers of a fuzzy matrix depends on its composition. The above Fuzzy Algebras ([0, 1], max, $\odot$) are fairly general subclass of dioïds where addition is the maximum of two real numbers and $\odot$ is an arbitrary $t$-norm [9]. The algebraic structures ([0, 1], max, min) (see, e.g. [4]) and ([0, 1], max, $\cdot$) are recognized as instances of dioïds, where "$\cdot$" is the standard product. Applications of dioïd and semiring structures, stressing links with Fuzzy Sets and emphasizing linear algebraic problems, non-classical path-finding problems and connections between dioïd structure and nonlinear analysis can be found in [9]. Lur et al. [19] has studied the powers of a max-arithmetic mean fuzzy matrix. It was showed that the powers of a max-arithmetic mean fuzzy matrix are always convergent. Moreover, the convex combination of the max-min and max-arithmetic mean powers of a fuzzy matrix are always convergent for all $0 \le \lambda < 1$ [18]. This result means that the major behavior of powers of a max-min fuzzy matrix shall change significantly and follow the property inherited from the max-arithmetic mean fuzzy matrix. The current paper is motivated by this result. We want to investigate what behavior of powers of a fuzzy matrix will happen when the max-arithmetic operator is replaced by max-product operator.

In this paper, we show that the powers of a $\otimes$ fuzzy matrix are asymptotical $p$-period if and only if the powers of $\bar{A}$ are $p$-periodic for all $0 < \lambda \le 1$. Thus as long as $\lambda \ne 0$, the limiting behavior of powers of this fuzzy matrix is same to that of a max product fuzzy matrix. This result is very interesting if we consider whether the max-min operation is "robust" or not. When the max-min operation is "perturbed" a little bit in sense of our convex combination (this may correspond to a $\lambda$ very close to 0), then the major behavior of powers of a max-min fuzzy matrix shall change significantly and follow the property inherited from the max product fuzzy matrix.

## 2. Preliminaries

Let $0 \le \lambda \le 1$ be given. The convex combination of the product and min operator $\otimes$ from $[0, 1] \times [0, 1]$ to $[0, 1]$ is defined by

$$a \otimes b = \lambda ab + (1-\lambda)\min\{a, b\},\ \text{ for all } a, b \in [0, 1].$$

It follows that $(a \otimes b) \otimes c$ is not necessary equal to $a \otimes (b \otimes c)$, that is, $\otimes$ is nonassociative. The product of $a_1 \otimes a_2 \otimes \cdots \otimes a_k$ is meant to be

$$a_1 \otimes a_2 \otimes \cdots \otimes a_k = ((((a_1 \otimes a_2) \otimes a_3) \otimes \cdots) \otimes a_k),$$

for all $a_1, a_2, \dots, a_k \in [0, 1]$. Define $a^1_\otimes = a$ and $a^k_\otimes = a^{k-1}_\otimes \otimes a$ for all $k \ge 2$. Let $a, b, c, d \in [0, 1]$ be given. It is obvious that $a \otimes b \le c \otimes d$ if $a \le c$ and $b \le d$. Recall that the convex combination of max-product and max-min composition for $A = [a_{ij}]$ is defined as