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Completeness of fuzzy normed linear space of all weakly fuzzy bounded linear operators ☆

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Abstract

In this note, it is proved that the fuzzy normed linear space $B(X, Y)$ of all weakly fuzzy bounded linear operators defined from a Felbin's type fuzzy normed linear space X to a Felbin's type fuzzy normed space Y is complete when Y is complete, and that Y is complete when $B(X, Y)$ is complete with some mild conditions on X.

Keywords: Fuzzy normed space; Weakly fuzzy bounded linear operator; Completeness

1. Introduction

In 1984, Katsaras [3] first introduced an idea of fuzzy norm on a linear space. In 1992, Felbin [2] defined a fuzzy norm by assigning a non-negative fuzzy real number rather than a real number to each element of a linear space. Xiao and Zhu [5,6] considered fuzzy norm in general form. Bag and Samanta [1] gave the definitions of weakly fuzzy bounded linear operators and strongly fuzzy bounded linear operators and their fuzzy norms. They proved that both the set $B(X, Y)^s$ of all strongly fuzzy bounded linear operators and the set $B(X, Y)$ of all weakly fuzzy bounded linear operators from a Felbin's type fuzzy normed linear space X into another Felbin's type fuzzy normed space Y are fuzzy normed linear spaces. They also got the Hahn–Banach theorem for strongly fuzzy bounded linear functionals.

In the theory of functional analysis, it is well known that the normed linear space $L(X', Y')$ of all bounded linear operators from a normed linear space X' into another normed linear space Y' is complete if and only if Y' is complete.

So it is interesting to investigate the relation between the completeness of the fuzzy normed linear space $B(X, Y)$ of all weakly fuzzy bounded linear operators defined from a Felbin's type fuzzy normed linear space X to a Felbin's type fuzzy normed space Y and the completeness of Y, and the relation between the completeness of the fuzzy normed linear space $B(X, Y)^s$ of all strongly fuzzy bounded linear operators defined from X to Y and the completeness of Y. In this note, we will prove that $B(X, Y)$ is complete when Y is complete, and that Y is complete when $B(X, Y)$ is complete with some mild conditions on X. And an example will be given to illustrate that the related theory of

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functional analysis is a corollary of the result of this paper. However, it needs a further study to research the relation between the completeness of $B(X,Y)^s$ and the completeness of Y.

As a preparation, in Section 2, we provide some useful definitions, notations and preliminary results. In Section 3, we investigate the relation between the completeness of $B(X,Y)$ and the completeness of Y.

2. Preliminaries

We denote the set of all fuzzy real numbers by F (see [5], slightly differ from [2]). If $\eta \in F$, η is called a positive fuzzy real number if $\eta(t) = 0$ for $t < 0$. We denote the set of all positive fuzzy real numbers by F^+. For $\eta \in F^+$ and $\alpha \in (0,1]$, α-level set $[\eta]_\alpha = \{t\colon \eta(t) \geqslant \alpha\}$ is a closed interval and we write it by $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$. For any real number r, $\bar{r}$ stands for the fuzzy real number which satisfies $\bar{r}(t) = 1$ if $t = r$ and $\bar{r}(t) = 0$ if $t \neq r$. 0 is the origin of linear space and zero.

A partial ordering "$\preccurlyeq$" in F is defined by $\eta \preccurlyeq \delta$ if and only if $\eta_\alpha^- \leqslant \delta_\alpha^-$ and $\eta_\alpha^+ \leqslant \delta_\alpha^+$ for all $\alpha \in (0,1]$. The strict inequality in F is defined by $\eta \prec \delta$ if and only if $\eta_\alpha^- < \delta_\alpha^-$ and $\eta_\alpha^+ < \delta_\alpha^+$ for all $\alpha \in (0,1]$.

According to Mizumoto and Tanaka [4] and Bag and Samanta [1], the arithmetic operations $\oplus$, $\otimes$ and $\oslash$ on $F^+ \times F^+$ are defined by

$$\eta \oplus \delta(t) = \sup_{s\in R} \min\{\eta(s), \delta(t-s)\}, \quad t \in R,$$

$$\eta \odot \delta(t) = \sup_{s\in R} \min\left\{\eta(s), \delta\left(\frac{t}{s}\right)\right\}, \quad t \in R,$$

$$\eta \oslash \delta(t) = \sup_{s\in R} \min\{\eta(st), \delta(s)\}, \quad t \in R, \delta \succ \bar{0}.$$

Proposition 2.1. *Let* $\eta, \delta \in F$ *and* $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$, $[\delta]_\alpha = [\delta_\alpha^-, \delta_\alpha^+]$, $\alpha \in (0,1]$. *Then*

$$[\eta \oplus \delta]_\alpha = [\eta_\alpha^- + \delta_\alpha^-, \eta_\alpha^+ + \delta_\alpha^+],$$

$$[\eta \odot \delta]_\alpha = [\eta_\alpha^- \delta_\alpha^-, \eta_\alpha^+ \delta_\alpha^+],$$

$$[\eta \oslash \delta]_\alpha = \left[\frac{\eta_\alpha^-}{\delta_\alpha^+}, \frac{\eta_\alpha^+}{\delta_\alpha^-}\right], \quad \delta_\alpha^- > 0, \quad \alpha \in (0,1].$$

For the case when $U = \vee(\max)$ and $L = \wedge(\min)$, the definition of a fuzzy norm on a linear space is given as following.

Definition 2.1. (See Bag and Samanta [1].) Let X be a linear space over the real number field R. Let $||\cdot|| : X \to F^+$ be a mapping satisfying:

(i) $\|x\| = \bar{0}$ iff $x = 0$;
(ii) $\|rx\| = \bar{r} \odot \|x\|$ for $x \in X$ and $r \in (-\infty, +\infty)$;
(iii) $\|x + y\| \preccurlyeq \|x\| \oplus \|y\|$, and
(A′) $x \neq 0 \Rightarrow \|x\|(0) = 0$.

Then $(X, \|\cdot\|)$ is called a fuzzy normed linear space and $\|\cdot\|$ is called a fuzzy norm on X.

Definition 2.2. (See Felbin [2] and Xiao and Zhu [5].) Let $(X, \|\cdot\|)$ be a fuzzy normed linear space. A sequence $\{x_n\}$ in X is said to converge to denoted by $\lim_{n\to\infty} x_n = x$ iff $\lim_{n\to\infty} \|x_n - x\| = \bar{0}$, i.e. $\lim_{n\to\infty} \|x_n - x\|_\alpha^- = \lim_{n\to\infty} \|x_n - x\|_\alpha^+ = 0$ for all $\alpha \in (0,1]$.

$\{x_n\}$ is called a Cauchy sequence if $\lim_{n,m\to\infty} \|x_n - x_m\| = \bar{0}$, i.e. $\lim_{n,m\to\infty} \|x_n - x_m\|_\alpha^- = \lim_{n,m\to\infty} \|x_n - x_m\|_\alpha^+ = 0$ for all $\alpha \in (0,1]$.

$(X, \|\cdot\|)$ is called a fuzzy Banach space if it is complete, i.e. every Cauchy sequence in X converges in X.