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A new fuzzy approximation method to Cauchy problems by fuzzy transform

A. Khastan [a,*], I. Perfilieva [b], Z. Alijani [a]

[a] *Department of Mathematics, Institute for Advanced Studies in Basic Sciences (IASBS), Zanjan 45137-66731, Iran*
[b] *University of Ostrava, Institute for Research and Applications of Fuzzy Modelling, NSC IT4Innovations, 30. dubna 22, 701 03 Ostrava, Czech Republic*

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Abstract

We present new numeric methods based on the first and second degree F-transform for solving the Cauchy problem. We show that they outperform the second order Runge–Kutta method especially when a right-hand function is oscillating and/or a solution is requested on a long interval. Moreover, a new numeric method to solve fuzzy initial value problem using the F-transform is presented. The suitability of the presented new methods is theoretically justified and illustrated on various examples.

Keywords: F-transform; Higher degree F-transform; Cauchy problem; Fuzzy initial value problem; Mid-point rule

1. Introduction

In this paper, we continue the study of fuzzy-based contributions to solutions of classical problems. In this connection, it is well-known that fuzzy methods successfully cope with approximation and interpolation problems, control of dynamic systems, solving differential equations and have applications in decision making, operation research and in general, data and image processing.

On the other hand, fuzzy methods are indispensable in handling ill-defined problems or those containing uncertainty in formulation. For example, in the lastly mentioned direction, many new techniques (the Hukuhara derivative, differential inclusion) were elaborated for solving fuzzy differential equations, see e.g. [4,6,7].

The proposed contribution is based on a special fuzzy method – the F(fuzzy)-transform, that can be successfully applied to both classical problems and those ones that contain fuzzy parameters. We will be focused on numeric methods for solving ordinary and fuzzy differential equations.

In 2003, the technique of F-transforms has been applied for solving ordinary differential equations with the purpose to show its potential in comparison with numerous classical techniques. A numeric method which generalizes the Euler

* Corresponding author.
E-mail addresses: khastan@iasbs.ac.ir, khastan@gmail.com (A. Khastan), irina.perfilieva@osu.cz (I. Perfilieva).

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method has been proposed in [10] for an ordinary Cauchy problem. This method (as well as its classical prototype) has the same accuracy and belong to class the order one methods.

In [10], the author came with the idea how the F-transform technique can be generalized from the case of constant components to the case of polynomial components. It has been proved that every polynomial component approximates a certain restriction of an original function and that as the degree of the polynomial increases, so does the quality of approximation. In [13], besides the general characterization of the F^m-transform, $m \geq 0$, more details were given to the case $m = 1$.

In the current contribution, we present three new numeric schemes for solving the ordinary Cauchy problem. All these schemes are based on the F-transform of various degrees. The first method (Mid-FT) improves the previously proposed [10] F-transform-based Euler method (denoted as Euler-FT) by the more efficient way of computation approximate derivatives (symmetric differences instead of directed ones). The two other methods (denoted as F^2-transform-based Schemes I and II) use the first and second degree F-transforms for a more accurate computation of the unknown function. All three methods provide better solutions than ordinary numeric methods (up to the second order Runge–Kutta one) on examples where the right-hand side is oscillating or the solution should be found on a long interval (that contains several oscillating parts with an increasing amplitude).

Moreover, we also discuss fuzzy differential equations in the form of the Cauchy problem with fuzzy valued function in its right-hand side and fuzzy initial condition (FIVP). This problem has been studied by many authors, and various numeric methods for its solution were proposed [1,4,7]. These methods are based on the assumption of Hukuhara differentiability or generalized Hukuhara differentiability. In the literature [1,5,8,9], they are known as the fuzzy Euler method, predictor–corrector method and Nyström method. Our contribution to solve FIVP consists in the application of the F-transform to the equivalent system of ODEs and in obtaining the corresponding Mid-point rule for a numeric solution.

The paper is organized as follows. In Section 2, we remind necessary basic concepts and prove new estimates (Lemma 2.12 and its corollary). In Section 3, we propose the F-transform-based method Mid-FT for the numeric solution of the ordinary Cauchy problem. In Section 4, new F^2-transform-based Schemes I and II for the ordinary Cauchy problem are presented. Numeric examples are discussed in Subsection 4.3. In Section 5, we present the Mid-point rule, based on the F-transform, to solve the fuzzy initial value problem. Finally, conclusions are given.

2. Preliminaries

In this section, we give some definitions and introduce the necessary notation which will be used throughout the paper, see for example [7,10,11,13]. In the first part (Subsection 2.1), the necessary facts from the theory of the F-transforms are given, including new estimates of coefficients of the 2nd order F-transform (Lemma 2.12) and a new local approximation of an original function by its 2nd order F-transform component (Corollary 2.13). In the second part (Subsection 2.2), we recall the notion of fuzzy number, introduce a space of fuzzy numbers and the notion of differentiability for a fuzzy-valued function.

2.1. The F-transform and its higher degree components

Definition 2.1. Let $x_1 < \ldots < x_n$ be fixed nodes within $[a, b]$, such that $x_1 = a$, $x_n = b$ and $n \geq 2$. We say that fuzzy sets $A_1, \ldots, A_n$, identified with their membership functions $A_1(x), \ldots, A_n(x)$ defined on $[a, b]$, form a fuzzy partition of $[a, b]$ if they fulfill the following conditions for $k = 1, \ldots, n$,

1. $A_k : [a, b] \to [0, 1]$, $A_k(x_k) = 1$;
2. $A_k(x) = 0$ if $x \notin (x_{k-1}, x_{k+1})$, where for the uniformity of the notation, we put $x_0 = a$ and $x_{n+1} = b$;
3. $A_k(x)$ is continuous;
4. $A_k(x)$, $k = 2, \ldots, n$, strictly increases on $[x_{k-1}, x_k]$ and $A_k(x)$, $k = 1, \ldots, n-1$, strictly decreases on $[x_k, x_{k+1}]$;
5. for all $x \in [a, b]$, $\sum_{k=1}^{n} A_k(x) = 1$.

The membership functions $A_1, \ldots, A_n$ are called *basic functions*. Let us remark that basic functions are specified by a set of nodes $x_1 < \ldots < x_n$ and the properties 1–5.