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Consistency and consensus improving methods for pairwise comparison matrices based on Abelian linearly ordered group

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Abstract

The aim of a valued pairwise comparison matrix is to derive the priority structure over a set of criteria (or alternatives) in decision making. The consistency and consensus of a pairwise comparison matrix should be measured and improved to avoid a misleading priority structure. The basic entries of a pairwise comparison matrix can be described in different forms; hence, different consistency and consensus methods should be developed for different types of matrices. To provide a general framework, the pairwise comparison matrix based on Abelian linearly ordered group is first introduced. A consistency index is defined by constructing the nearest consistent pairwise comparison matrix from an inconsistent one, and two consistency improving methods are introduced. A group pairwise comparison matrix is derived, a consensus index of individual pairwise comparison matrices is defined and two consensus improving methods are developed by introducing a general aggregation operator based on Abelian linearly ordered group. The proposed consistency and consensus methods are convergent and can provide a general framework for existing methods.

Keywords: Decision making; Pairwise comparison matrix; Abelian linearly ordered group; Consistency; Consensus

1. Introduction

The pairwise comparison (PC) matrix [19] is an important technique in decision making that has been widely applied in areas, such as risk assessment, total quality management, R&D project selection, resource allocation, low carbon supply chain assessment, etc. The PC matrix is used to derive the priority structure over a set of criteria (or alternatives) by comparing the elements inside each pair. For example, a supply chain can be evaluated by several criteria, such as cost, quality, lead time and carbon emission. Since a supply chains is composed of a lot of enterprises, each enterprise can be considered as a decision maker and can construct a PC matrix by comparing the criteria inside each pair to derive the priority over the criteria. Saaty [18] proposed a method in the setting of fuzzy sets by

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structuring the functions of a system hierarchically in a multiple objective framework. Exploring the interface between hierarchies, multiple objectives and fuzzy sets PC matrices have different forms, which include the following: the multiplicative PC matrix [19], which expresses the preference ratio of each pair of criteria (or alternatives), the additive PC matrix [13], which expresses the preference difference, and the fuzzy PC matrix [20], which expresses the distance from the indifference. However, these different types of PC matrices have several common points, such as their entries a_{ij} express the degree to which x_i is prior to x_j, and a condition of reciprocity is always assumed (the entry a_{ij} can be exactly expressed by a_{ji}). By introducing the Abelian linearly ordered group, Cavallo and D'Apuzzo [6] defined a general framework that can unify all kinds of PC matrices.

Consistency and consensus should be measured and improved to avoid a misleading priority structure. Consistency measures the degree of agreement among the preference values provided by an individual decision maker, while consensus measures the degree of agreement among decision makers on the solution to the problem. Different consistency or consensus methods should be defined for different types of PC matrices. Different types of PC matrices have common points, so do the consistency or consensus methods. Motivated by the general theoretical framework [4–6], this study focuses on the general unified framework of the consistency and consensus methods for different types of PC matrices based on Abelian linearly ordered group.

The remainder of this paper is organized as follows: Several preliminary studies are introduced in Section 2. The properties of PC matrices are investigated based on Abelian linearly ordered group (alo-group for short) in Section 3. A method to derive a consistent PC matrix from an inconsistent one is also provided in this section. The consistency index and two methods are presented to improve the consistency of PC matrices in Section 4. A method is proposed to aggregate individual PC matrices based on alo-group in Section 5. The consensus index and two methods are also provided to improve the consensus of individual PC matrices. Concluding remarks are given in Section 6. The structure of the paper is illustrated in Fig. 1 (see Appendix A).

2. PC matrix based on alo-group

This section mainly introduces some preliminaries about the PC matrix based on Abelian linearly order group.

To obtain the priority structure over a set of subjects $X = \{x_1, x_2, \cdots, x_n\}$, it is usual to compare elements inside each pair and construct a relation $\mathrm{A} : (x_i, x_j) \in X^2 \to a_{ij} = \mathrm{A}(x_i, x_j) \in G \subseteq R$ represented by the pairwise comparison (PC) matrix $A = (a_{ij})_{n\times n}$, where a_{ij} expresses the degree to which subject x_i is preferred to subject x_j. The entry a_{ij} can be described in different forms according to the preferences of decision makers; these forms include the multiplicative PC matrix [19], the additive PC matrix [13], and the fuzzy PC matrix [20]. Cavallo and D'Apuzzo [6] defined the following concepts to unify these different types of PC matrices.

Let G be a nonempty set, $\odot : G \times G \to G$ a binary operation on G, and $\leq$ a total weak order on G. Then $\Omega = (G, \odot, \leq)$ is called an Abelian linearly ordered group (alo-group for short) if and only if $(G, \odot)$ is an Abelian group and

$$a \leq b \quad \Rightarrow \quad a \odot c \leq b \odot c \tag{1}$$

Let $\Omega = (G, \odot, \leq)$ be an alo-group. Then the following symbols are presented:

e: the identity of Ω,
$x^{[-1]}$: the symmetric of $x \in G$ with respect to $\odot$,
$\div$: the inverse operation of $\odot$ defined by $a \div b = a \odot b^{[-1]}$,
$<$: the strict simple order defined by "$x < y \leftrightarrow x \leq y$ and $x \neq y$",
$\geq$ *and* $>$: the opposite relation of $\leq$ and $<$, respectively.

Then $b^{[-1]} = e \div b$, $(a \odot b)^{[-1]} = a^{[-1]} \odot b^{[-1]}$, $(a \div b)^{[-1]} = b \div a$.

Let $\Omega = (G, \odot, \leq)$ be an alo-group. For a positive integer n, the (n)-power $x^{[n]}$ of $x \in G$ is defined by

$$\begin{cases} x^{[1]} = x \\ x^{[n]} = \bigodot_{i=1}^{n} x_i, \quad x_i = x, \ \forall i = 1, \cdots, n, \ \text{for } n \geq 2 \end{cases}$$